北京大学中国经济研究中心
China Center for Economic Research

讨论稿系列
Working Paper Series

E2022017 2022-10-07

Optimal Contingent Delegation

Tan Gan、Ju Hu、Xi Weng

Abstract

This paper investigates a two-agent mechanism design problem without transfers, where the principal must decide one action for each agent. In our framework, agents only care about their own adaptation, and any deterministic dominant incentive compatible decision rule is equivalent to contingent delegation: the delegation set offered to one agent depends on the other's report. By contrast, the principal cares about both adaptation and coordination. We provide sufficient conditions under which contingent interval delegation is optimal and solve the optimal contingent interval delegation under fairly general conditions. Remarkably, the optimal interval delegation is completely determined by combining and modifying the solutions to a class of simple single-agent problems, where the other agent is assumed to report truthfully and choose his most preferred action.

Keywords: Dominant strategy mechanism design, contingent delegation, adaptation, coordination

JEL Classification: D23, D82, L23, M11

Optimal Contingent Delegation*

Tan Gan[†] Ju Hu[‡] Xi Weng[§]

First version: June 4, 2021
This version: October 5, 2022

Abstract

This paper investigates a two-agent mechanism design problem without transfers, where the principal must decide one action for each agent. In our framework, agents only care about their own adaptation, and any deterministic dominant incentive compatible decision rule is equivalent to contingent delegation: the delegation set offered to one agent depends on the other's report. By contrast, the principal cares about both adaptation and coordination. We provide sufficient conditions under which contingent interval delegation is optimal and solve the optimal contingent interval delegation under fairly general conditions. Remarkably, the optimal interval delegation is completely determined by combining and modifying the solutions to a class of simple single-agent problems, where the other agent is assumed to report truthfully and choose his most preferred action.

KEYWORDS: Dominant strategy mechanism design, contingent delegation, adaptation, coordination

JEL CLASSIFICATION: D23, D82, L23, M11

*We thank the editor, Tilman Börgers, an associate editor, and three anonymous referees for their insightful comments and suggestions, which have greatly improved the paper. We are very grateful for helpful comments from Dirk Bergemann, Marina Halac, Johannes Hörner, Matthew Knudson, Heng Liu, and Shuo Liu. We also thank the seminar participants at Peking University, Yale, YES at Columbia, CCER Summer Institute, and the 2022 Conference on Mechanism and Institution Design for helpful comments. Ju Hu and Xi Weng acknowledge the financial support of the NSFC (Grant Nos. 72273007, 71973002, 72192843 and 72225001). All remaining errors are ours.

[†]Department of Economics, Yale University; Email: tan.gan@yale.edu.

[‡]China Center for Economic Research, National School of Development, Peking University; Email: juhu@nsd.pku.edu.cn.

[§]Guanghua School of Management, Peking University; Email: wengxi125@gsm.pku.edu.cn.

1 Introduction

This paper presents an analysis of a mechanism design problem with a principal (she) and two agents (he), and without monetary transfers. The principal needs to make two decisions, one for each agent, but the relevant information is dispersed between the agents. While each agent only cares about the decision for himself, the principal also cares about the interactions of the two decisions.

An application of our analysis is to the delegation problem within multidivisional organizations. As pointed out by Roberts (2004) and Alonso et al. (2008), multidivisional organizations exist primarily to coordinate the activities of their divisions. Coordinated decision making by the headquarters manager requires aggregation of the relevant information, which is usually dispersed among the individual division managers as they are best informed of their local conditions. But there is a conflict of interest between the headquarters manager, who cares more about coordination, and the division managers, who care more about adaptation: more coordinated decisions are less adapted to the local conditions of each division. In such an environment, how should the headquarters manager delegate to the division managers to reflect the trade-off between adaptation and coordination? This question is unexplored in the prior literature on authority allocation within multidivisional organizations.[1] Our paper fills the gap as a direct application of our main result can shed light on the optimal design of delegation rules.

Formally, each of the two agents in our model has a quadratic-loss payoff function that only depends on his own state and the decision for him. Each agent's most preferred decision is equal to his state. By contrast, the principal's payoff function consists of three additively separable components. Two of them are called *adaptation payoffs*, which represent her potentially different preferences over each agent's decision and the corresponding state. In general, we allow incentive misalignment in the sense that these payoffs are different from the agents' ones. The third component is a supermodular function that only depends on the agents' actions. The complementarity of the two actions captures the principal's coordination motive: if one agent makes a higher decision, she would like the other agent to make a higher decision too. Thus, we refer to this component as the principal's *coordination payoff*.

The principal can commit to any deterministic dominant strategy incentive compatible mechanism, which can be implemented by a *contingent delegation* mechanism. In such a mechanism, agents report their states to the principal and then the princi-

[1] See further elaboration in the literature review.

pal offers each agent a delegation set that depends on the other agent's report. After reporting and receiving his own delegation set, each agent chooses his favorite action from it. Our goal is to understand the principal's optimal contingent delegation.

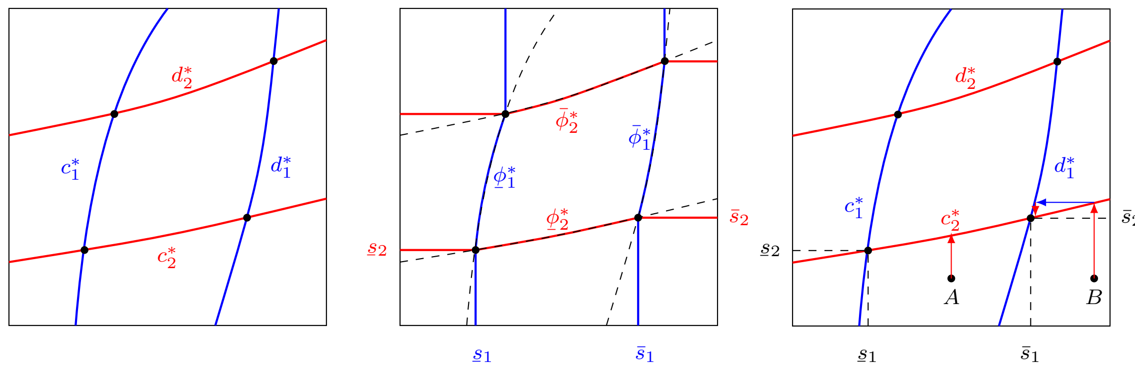
To see the main problems faced by the principal in her design, consider the previously mentioned coordination problem in multidivisional organizations. If the headquarters manager only cared about whether the decisions of the local divisions were adapted to their local conditions and had no coordination concern at all, then she could simply grant full discretion and delegate all the decision rights to the local divisions, since their interests were perfectly aligned. However, this decision rule should not be optimal in the presence of the coordination motive, as the local divisions' fully adapted decisions may not be well coordinated, leading to a large coordination loss. To mitigate such miscoordination, the headquarters manager can give less discretion to the local divisions. By ruling out some decisions for a division, she can induce this division to coordinate with the other one at the cost of reduced adaptation of this division. Thus, the optimal level of discretion for each division must trade off the cost from reduced adaptation against the benefit from better coordination. The difficulty here is that each division's trade-off depends on the other division's decision, which in turn is determined by the discretion the other division is granted. Therefore, the optimal design must resolve both divisions' trade-offs jointly.

Our first main result, Theorem 1, sheds light on how these trade-offs are resolved jointly at the optimum. It characterizes the optimal *contingent interval delegations*, under which the contingent delegation sets that the principal offers to the agents are always intervals. We construct the optimal solution via a "two-step procedure." The first step treats each agent's trade-off separately, while the second step deals with the joint design problem.

In the first step, we consider the principal's optimal interval contingent delegation problem for agent i, assuming that agent $-i$ is granted full discretion. This involves a series of simple single-agent problems, in each of which the principal determines agent i's delegation interval to maximize the expected sum of her adaptation payoff from agent i and her coordination payoff, given that agent $-i$'s state is s_{-i} and he chooses $a_{-i} = s_{-i}$. We assume that for each s_{-i}, the optimal interval $[c_i^*(s_{-i}),\, d_i^*(s_{-i})]$ is uniquely determined and non-degenerate.[2] Both boundary functions c_i^* and d_i^* are nondecreasing in s_{-i}, because the principal would like agent i to take higher action to coordinate better with agent $-i$ when $-i$ takes a higher action. We refer to the

[2]This is condition U in Section 3.2. Sufficient conditions on the model primitives are provided in Lemma 3.

pair of functions (c_i^*, d_i^*) as the *unilaterally constrained delegation rule* for agent i, because it is obtained by assuming that agent $-i$ is never constrained. Panel (a) in Figure 1 provides an illustration of the unilaterally constrained delegation rules for both agents. The square is the s_1, s_2-plane.[3] The blue and red curves represent the unilaterally constrained delegation rules for agents 1 and 2, respectively.



(a) Unilaterally constrained delegation rules

(b) Optimal contingent interval delegation

(c) Intuitive modification

Figure 1: Optimal contingent interval delegation

These two unilaterally constrained delegation rules together give the principal a contingent interval delegation $((c_1^*, d_1^*), (c_2^*, d_2^*))$. But intuitively it is not optimal, precisely because it neglects the joint design problem: changing from full discretion to delegation rule (c_{-i}^*, d_{-i}^*) changes agent $-i$'s behavior, which in turn affects agent i's coordination problem and makes (c_i^*, d_i^*) for agent i suboptimal. To see this, consider, for example, a sufficiently low s_2 so that action s_2 is never available to agent 2 under (c_2^*, d_2^*). Under this contingent delegation rule, agent 2's action will always be higher than what he would take under full discretion, i.e., s_2. This implies that the delegation interval $[c_1^*(s_2),\, d_1^*(s_2)]$ for agent 1 is no longer optimal, because the principal would like to move this interval upward for better coordination.

Nonetheless, we resolve this issue in the second step by modifying $((c_1^*, d_1^*), (c_2^*, d_2^*))$, under the additional assumption that c_1^* and d_1^* intersect c_2^* and d_2^*, respectively, only once in the s_1, s_2-plane, as is the case in panel (a).[4] Theorem 1 states that an

[3] For ease of exposition, we assume that the state space and the action space are the same for both agents.

[4] This is condition R in Section 3.2. Sufficient conditions on the model primitives are provided in Lemma 4.

optimal contingent interval delegation is immediately obtained by bounding the unilaterally constrained delegation rules with the intersections. The resulting contingent delegation is illustrated in panel (b).[5] The curve $\underline{\phi}_i^*$ is the lower bound and $\bar{\phi}_i^*$ is the upper bound so that the delegation interval for agent i when $-i$ reports s_{-i} is $[\underline{\phi}_i^*(s_{-i}),\, \bar{\phi}_1^*(s_{-i})]$.

To gain some intuition on the construction of the optimal mechanism, consider again the example where s_2 is sufficiently low so that action s_2 is never available to agent 2 under $(c_2^*(s_1), d_2^*(s_1))$. We employ an iterative process of modifications in this case by first increasing agent 2's action, a_2, to the lower bound $c_2^*(s_1)$. This change of a_2 implies that the delegation interval $[c_1^*(s_2),\, d_1^*(s_2)]$ for agent 1 is no longer optimal, and intuitively we change it to $[c_1^*(c_2^*(s_1)),\, d_1^*(c_2^*(s_1))]$. If s_1 is contained in the interval $[c_1^*(c_2^*(s_1)),\, d_1^*(c_2^*(s_1))]$, we can stop further modifications and let $a_1 = s_1$. This is how we use the arrow to modify point A in panel (c) in Figure 1. But if s_1 is outside the interval, we need to change a_1 to the boundary, and this triggers further modifications of agent 2's delegation interval. This iterative process continues until it converges to $(\bar{s}_1, \bar{s}_2)$, as illustrated by the arrow starting from point B in panel (c). Consequently, the optimal delegation is flat over the corner.

Our second main result, Theorem 2, establishes sufficient conditions for the optimal contingent interval delegation in Theorem 1 to be optimal among all the contingent delegation mechanisms. These sufficient conditions are expressed in terms of the principal's adaptation and coordination payoffs and the state distributions. A more general result, which provides sufficient conditions for any given contingent interval delegation to be optimal and on which Theorem 2 is based, is also provided in Theorem 3 in the appendix. It extends the main sufficiency result in Amador and Bagwell (2013) to our two-agent setting.

Finally, we apply the above general results to study the previously mentioned optimal design problem within a multidivisional organization. Under the quadratic-loss specification of the principal's payoff function and log-concavity of the state distributions, all the conditions for Theorems 1 and 2 are satisfied. Therefore, the optimal contingent interval delegation we found in Theorem 1 is indeed an optimal mechanism. Due to the simple structure of this optimal contingent interval delegation, a set of intuitive comparative statics results are easily obtained. For one example, if coordination becomes more important to the principal, then both divisions will receive less discretion. For another example, if one division becomes more important to the principal, then this division must be better off in that it will be granted larger

[5] The dashed curves correspond to the unilaterally constrained delegation rules.

discretion. But the other division will suffer as it will receive less discretion.

Related literature. Our work relates to two main strands of the literature. The first is the research on mechanism design without contingent transfers. In the single-agent setting, it is well known that such a problem is equivalent to the delegation problem. Holmström (1977, 1984) was the first to pose the general class of delegation problems. Since then, a number of other researchers, including Melumad and Shibano (1991), Martimort and Semenov (2006), Alonso and Matouschek (2008), Amador and Bagwell (2013), and Amador et al. (2018), have studied and characterized the solution to the single-agent delegation problem under various assumptions on the preferences and state distributions. This literature places particular emphasis on the optimality of interval delegation since it is the most natural form and is commonly observed in reality. By focusing on dominant strategy incentive compatible mechanisms, we establish a similar equivalence between mechanism design and delegation in our general framework with two actions and two agents.

To our knowledge, Alonso et al. (2014) were the first to study optimal mechanism design without contingent transfers in an environment with multiple actions and multiple agents. In their model, a principal allocates limited resources to three agents. Two of them are privately informed of their own ideal demand, and the ideal demand of the third agent is known to the principal. Agents are biased only in one direction so only a cap will be used in the optimal unilaterally constrained delegation rules and consequently in the optimal mechanism. Our analysis points out that the decomposition result holds with general functional form and, in particular, in the presence of biases in both directions.[6] There are two other papers studying optimal non-monetary design with two agents and one action: Martimort and Semenov (2008) and Fuchs et al. (2022). Because the policy chosen by the principal is only one-dimensional, the models are more closely related to the single-agent case. For example, Fuchs et al. (2022) point out that when agents' type spaces are disjoint, the principal might find it optimal to delegate the decision right to just one agent.

The second strand studies authority allocation within multidivisional organizations. Similar to our setting, this literature assumes that multiple decisions must be coordinated and the relevant information for decision making is horizontally dispersed. However, related studies including Alonso et al. (2008), Rantakari (2008),

[6] When the principal has enough resources so that it is always feasible to meet the two privately informed agents' ideal demands, their model becomes a special case of ours after substituting the allocation of the third agent by the resource constraint. See also footnote 8.

Dessein et al. (2010), Friebel and Raith (2010), and Li and Weng (2017), assume a lack of commitment power in the sense that the organization can commit only to an ex ante allocation of decision rights, and explore strategic communication equilibria given an authority allocation mechanism in such settings.[7] For example, Alonso et al. (2008) compare the efficiency of centralization, in which case the division managers communicate vertically with the headquarters manager who will make the decisions, and decentralization, in which case the division managers who will make their own individual decisions communicate horizontally with each other. While all these papers study equilibria under certain exogenously given mechanisms, we apply our main result to this environment to investigate the optimal mechanism under full commitment power. To the best of our knowledge, our paper is the first to study the optimal design of delegation rules to reflect the trade-off between adaptation and coordination in multidivisional organizations, although admittedly our framework simplifies the setup by assuming that division managers only care about themselves, while papers such as Alonso et al. (2008) allow agents also to care about coordination (just to a lesser degree).

The rest of the paper is organized as follows. Section 2 describes the model. Section 3 contains the analysis and our main results. In Section 4, we apply our general results to the multidivisional organization problem. Section 5 concludes. The proofs for Section 3 are deferred to the appendix. The proofs for Section 4 can be found in the online appendix.

2 Model Setting

There are one principal and two agents. The principal needs to make one decision for each agent. She can commit to a deterministic decision rule but is unable to commit to contingent transfers.

Preferences: The principal's and the agents' payoffs depend on both the decision and the state of the world. A decision consists of a pair of actions, $a_1 \in [0,1]$ for agent 1 and $a_2 \in [0,1]$ for agent 2. A state of the world is a pair $(s_1, s_2) \in [0,1]^2$, with the interpretation that s_i is agent i's state.

Agent i's payoff only depends on his own state s_i and the decision a_i for him. In particular, we assume that i's payoff function takes the quadratic loss form $v_i(a_i, s_i) =$

[7] There are also related models where the communication is not strategic. See, for instance, Aoki (1986), Hart and Moore (2005), Dessein and Santos (2006), and Cremer et al. (2007).

$\frac{1}{2}(a_i - s_i)^2$. That is, each agent always wants the decision for him to be as close to his state as possible.

The principal, in contrast, cares about both decisions for the two agents and their states. Her payoff function is denoted by $u(a_1, a_2, s_1, s_2)$. Throughout the paper, we assume that u takes the following form:

$$u(a_1, a_2, s_1, s_2) \equiv u_0(a_1, a_2) + u_1(a_1, s_1) + u_2(a_2, s_2).$$

All the components u_0, u_1 and u_2 are twice continuously differentiable and concave in (a_1, a_2). The principal's payoff is a generalization of the literature on adaptation versus coordination in multidivisional organizations, for example, Alonso et al. (2008), Rantakari (2008), Alonso et al. (2014), and Li and Weng (2017). In particular, $u_i(a_i, s_i)$ for $i = 1, 2$ can be viewed as an adaptation payoff that measures how a_i is adapted to state s_i. This includes the case where $u_i(a_i, s_i)$ is proportional to $-(a_i - s_i)^2$, which is the specification in our application in Section 4. More generally, we can allow incentive misalignment in the sense that the principal values the adaptation payoff in a way that is different from the agents. Following the literature, we introduce $u_0(a_1, a_2)$ as a coordination payoff that measures how well the decisions are coordinated. For this interpretation, we assume that u_0 is supermodular so that the two decisions are complementary to each other.[8] This additively separable form of the principal's payoff function makes the interaction between the two decisions state-independent. As we shall see, this assumption implies that agent $-i$'s state s_{-i} has no direct effect on the design of agent i's decision. Its effect is only indirect through its effect on agent $-i$'s decision.

Information: Agent i perfectly knows his own state s_i, but not the other agent's state s_{-i}. The principal knows neither s_1 nor s_2. She believes that s_1 and s_2 are independently distributed over the interval $[0, 1]$, with cumulative distribution function F_1 and F_2. We assume that state s_i has full support and continuous density f_i. Because we focus on mechanisms that are dominant strategy incentive compatible, we do not need to specify each agent's belief about the other agent's state. Even the principal's prior belief can be completely subjective. It need not reflect the true distribution of the states.

[8] Our analysis can also deal with the case where u_0 is submodular by the simple trick of changing variables in (3): $\tilde{s}_2 = 1 - s_2$ and $\tilde{a}_2 = 1 - a_2$. In this way, $\tilde{u}_0(a_1, \tilde{a}_2) \equiv u_0(a_1, 1 - a_2)$ is supermodular in $(a_1, \tilde{a}_2)$. In this case, all the conditions that we impose later on u_0 should be understood as conditions on $\tilde{u}_0$.

Mechanism design problem: Throughout this paper we focus on deterministic mechanisms that are dominant strategy incentive compatible (DSIC), which requires that reporting truthfully is always optimal regardless of the other agent's report. Invoking the revelation principle, we can focus on direct mechanism (a_1, a_2), where each a_i is a measurable function that maps the reported states $(s_1, s_2) \in [0,1]^2$ to the action $a_i(s_1, s_2) \in [0,1]$ for agent i.[9] The design problem can be expressed as:

$$\begin{aligned} \max_{(a_1,a_2)} \quad & \int_0^1 \int_0^1 u\left(a_1(s_1,s_2), a_2(s_1,s_2), s_1, s_2\right) \mathrm{d}F_1(s_1)\mathrm{d}F_2(s_2) \\ \text{s.t.} \quad & v_i(a_i(s_i,s_{-i}), s_i) \geq v_i(a_i(\hat{s}_i, s_{-i}), s_i) \quad \forall i, s_i, \hat{s}_i, s_{-i}. \end{aligned} \tag{1}$$

3 Optimal Mechanism

In this section, we solve the principal's mechanism design problem (1) under some additional conditions. Section 3.1 introduces the notion of contingent delegation mechanisms, and establishes its equivalence to DSIC mechanisms in out setting. Sections 3.2 - 3.5 focus on contingent interval delegations, in which the delegation set offered to each agent is always an interval, and find an optimal contingent interval delegation. Section 3.6 provides conditions for this optimal contingent interval delegation to be optimal among all DSIC mechanisms.

3.1 Contingent delegation mechanisms

In single-agent settings, it is well known that the principal's direct mechanism design problem is equivalent to the delegation problem where the principal offers the agent a delegation set, from which the agent chooses his most preferred action (Holmström (1977, 1984), Melumad and Shibano (1991), Alonso and Matouschek (2008)). The following lemma essentially establishes a similar equivalence in our two-agent setting.

Lemma 1. *A direct mechanism (a_1, a_2) is a DSIC mechanism if and only if there exist closed-valued correspondences $D_i : [0,1] \rightrightarrows [0,1]$ for $i = 1,2$ such that, for all i, s_i, and s_{-i},*

$$a_i(s_i, s_{-i}) \in \underset{a_i' \in D_i(s_{-i})}{\arg\max}\ v_i(a_i', s_i). \tag{2}$$

[9] The revelation principle for the deterministic DSIC mechanisms holds if DSIC means each agent's report best replies to any strategies of other agents (in contrast to the definition of ex-post mechanisms). See Jarman and Meisner (2017) for details.

Lemma 1 states that any DSIC mechanism (a_1, a_2) is equivalent to a *contingent delegation mechanism* (D_1, D_2). In such a mechanism, the agents report their states to the principal. Instead of making decisions for the agents according to (a_1, a_2), the principal offers each agent i a delegation set $D_i(s_{-i})$, which is contingent on $-i$'s report and from which i is free to choose his favorite action. In this mechanism, every agent is willing to report truthfully because his payoff is completely determined by his own action. Equation (2) then states that the same decisions will be implemented under the DSIC mechanism and this corresponding contingent delegation mechanism.

In single-agent settings, interval delegation, where the principal offers an interval as the delegation set, is the most salient class of delegation mechanisms. This notion can also be naturally generalized to the current two-agent setting. A contingent delegation mechanism (D_1, D_2) is a *contingent interval delegation* if there exist measurable functions $\underline{\phi}_1, \bar{\phi}_1, \underline{\phi}_2, \bar{\phi}_2 : [0,1] \to [0,1]$ such that, for all i, $\underline{\phi}_i \leq \bar{\phi}_i$ and

$$D_i(s_{-i}) = [\underline{\phi}_i(s_{-i}),\, \bar{\phi}_i(s_{-i})], \quad \forall s_{-i} \in [0,1].$$

In such a mechanism, the delegation set that the principal offers to each agent is always an interval, and this interval varies with the other agent's report. From now on, we directly write this contingent interval delegation as (ϕ_1, ϕ_2), where $\phi_i = (\underline{\phi}_i, \bar{\phi}_i)$ is referred to as the *interval delegation rule* for agent i.

For $i = 1, 2$ and $0 \leq c \leq d \leq 1$, define

$$\sigma_i(s_i; c, d) \equiv \begin{cases} c, & \text{if } s_i < c, \\ s_i, & \text{if } c \leq s_i \leq d, \\ d, & \text{if } s_i > d. \end{cases}$$

Given agent i's quadratic-loss payoff function, $\sigma_i(s_i; c, d)$ is just i's most preferred decision at state s_i, when he is restricted to choose from the interval $[c, d]$. Given any contingent interval delegation (ϕ_1, ϕ_2), the corresponding DSIC mechanism, denoted by $(\sigma_1^{\phi_1}, \sigma_2^{\phi_2})$, is then given by[10,11]

$$\sigma_i^{\phi_i}(s_i, s_{-i}) \equiv \sigma_i(s_i; \underline{\phi}_i(s_{-i}), \bar{\phi}_i(s_{-i})),\ \forall i, s_i, s_{-i}.$$

[10]Measurability of $\sigma_i^{\phi_i}$ is guaranteed by measurability of ϕ_i.

[11]Conversely, any DSIC mechanism (a_1, a_2) that is continuous in one's own state, i.e., a_i is continuous in s_i for $i = 1, 2$, is equivalent to the contingent interval delegation defined by $\underline{\phi}_i(s_{-i}) = a_i(0, s_{-i})$ and $\bar{\phi}_i(s_{-i}) = a_i(1, s_{-i})$ for $i = 1, 2$. This is analogous to the well-known result in the single-agent delegation literature that a direct mechanism is equivalent to an interval delegation if and only if it is continuous.

3.2 Unilaterally constrained delegation rule

By Lemma 1, solving the principal's DSIC mechanism design problem (1) is equivalent to finding out the principal's optimal contingent delegation. For this, we first restrict our attention to contingent interval delegations and characterize the optimal contingent interval delegation (Theorem 1). Then, we show that under certain conditions, this optimal contingent interval delegation is optimal among all contingent delegations (Theorem 2).

The design of optimal contingent interval delegation can be written as

$$\max_{(\phi_1,\phi_2)} \int_0^1 \int_0^1 u\left(\sigma_1^{\phi_1}(s_1,s_2), \sigma_2^{\phi_2}(s_1,s_2), s_1, s_2\right) \mathrm{d}F_1(s_1)\mathrm{d}F_2(s_2), \tag{3}$$
$$\text{s.t. } \underline{\phi}_i(s_{-i}) \leq \bar{\phi}_i(s_{-i}), \quad \forall s_{-i}.$$

To solve this problem, we need to impose two additional conditions and introduce a special interval delegation rule for each agent. The basic purpose of doing so is to decompose the principal's design problem into two classes of single-agent delegation problems. These additional conditions will guarantee that the solutions to these single-agent delegation problems are nicely behaved. As Theorem 1 will show, a certain modification of the solutions to these single-agent problems becomes an optimal contingent interval delegation.

Suppose agent $-i$'s state is s_{-i} and he chooses his most preferred action $a_{-i} = s_{-i}$. Given agent $-i$'s behavior, consider the principal's optimal interval delegation problem for agent i. We can write it as

$$\max_{0\leq c\leq d\leq 1} \int_0^1 \left[u_0(\sigma_i(s_i;c,d), s_{-i}) + u_i(\sigma_i(s_i;c,d), s_i)\right] \mathrm{d}F_i(s_i). \tag{4}$$

By continuity of u_0 and u_i, an optimal solution to (4) always exists. The first condition we impose requires that the optimal delegation interval for this single agent problem be always unique and non-degenerate.

Condition U. *For every $s_{-i} \in [0,1]$, there is a unique solution $(c_i^*(s_{-i}), d_i^*(s_{-i}))$ to (4). It satisfies $c_i^*(s_{-i}) < d_i^*(s_{-i})$.*

Sufficient conditions on the payoff and distribution functions for condition U to hold are provided in Section 3.5. Viewing both c_i^* and d_i^* as boundary functions, (c_i^*, d_i^*) forms a delegation rule for agent i. It is indeed the principal's optimal interval delegation rule for agent i if agent $-i$ is always free to choose his most preferred action. For this reason, we refer to (c_i^*, d_i^*) as the *unilaterally constrained delegation rule for agent i.*

Condition U and supermodularity of u_0 give us two basic properties of the unilaterally constrained delegation rules.

Lemma 2. *Under condition U, both c_i^*, $d_i^* : [0,1] \to [0,1]$ are continuous and increasing, for $i = 1, 2$.*

Continuity is standard. Monotonicity comes from complementarity between the two actions under supermodularity of u_0. When $-i$ takes a higher action, the principal would like i to take a higher action as well. Hence, both the lower and upper bounds of the optimal delegation interval for i increase.

The second condition is a regularity condition for the two agents' unilaterally constrained delegation rules.

Condition R. *In the s_1, s_2-plane, the graphs of c_1^* and d_1^* intersect those of c_2^* and d_2^* only once, respectively.*

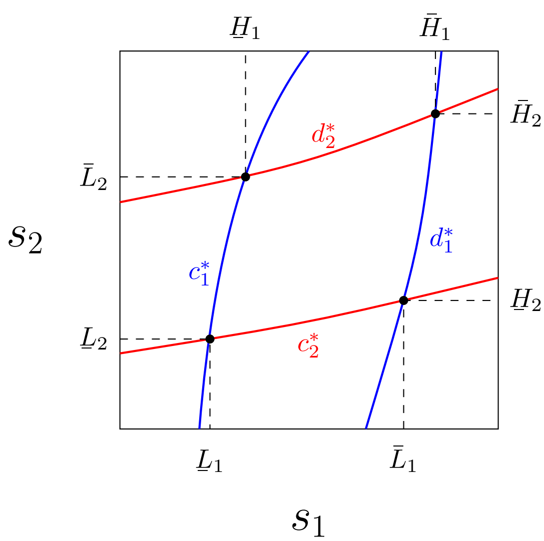


Figure 2: Unilaterally constrained delegation rules

Similar to condition U, sufficient conditions on the primitives for condition R are provided in Section 3.5. Figure 2 provides an illustration of typical pairs of unilaterally constrained delegation rules that satisfy condition R, which requires that any red curve and blue curve intersect only once. There are in total four intersections. We carefully label them in the graph and will follow this notation throughout the paper.

3.3 Optimal contingent interval delegation

Based on the unilaterally constrained delegation rules, we are now ready to state our first main result. We say that a contingent interval delegation (ϕ_1, ϕ_2) is increasing if all the boundary functions $\underline{\phi}_1, \bar{\phi}_1, \underline{\phi}_2$, and $\bar{\phi}_2$ are increasing. For example, $(c_1^*, d_1^*, c_2^*, d_2^*)$ is increasing according to Lemma 2. Let $\mathcal{M}$ be the set of all increasing contingent interval delegations. The following theorem constructs an optimal contingent interval delegation by modifying the unilaterally constrained delegation rules in a certain way according to their intersections. Moreover, this optimal contingent interval delegation is in $\mathcal{M}$, and it is essentially unique in $\mathcal{M}$.

Theorem 1. *Suppose conditions U and R hold. Denote the intersection of c_1^* and c_2^* by $(\underline{L}_1, \underline{L}_2)$, that of c_1^* and d_2^* by $(\underline{H}_1, \bar{L}_2)$, that of d_1^* and c_2^* by $(\bar{L}_1, \underline{H}_2)$, and that of d_1^* and d_2^* by $(\bar{H}_1, \bar{H}_2)$. For $i = 1, 2$, define*

$$\underline{\phi}_i^*(s_{-i}) \equiv \begin{cases} \underline{L}_i, & \text{if } s_{-i} \in [0,\ \underline{L}_{-i}], \\ c_i^*(s_{-i}), & \text{if } s_{-i} \in (\underline{L}_{-i},\ \bar{L}_{-i}), \\ \underline{H}_i, & \text{if } s_{-i} \in [\bar{L}_{-i},\ 1], \end{cases} \tag{5}$$

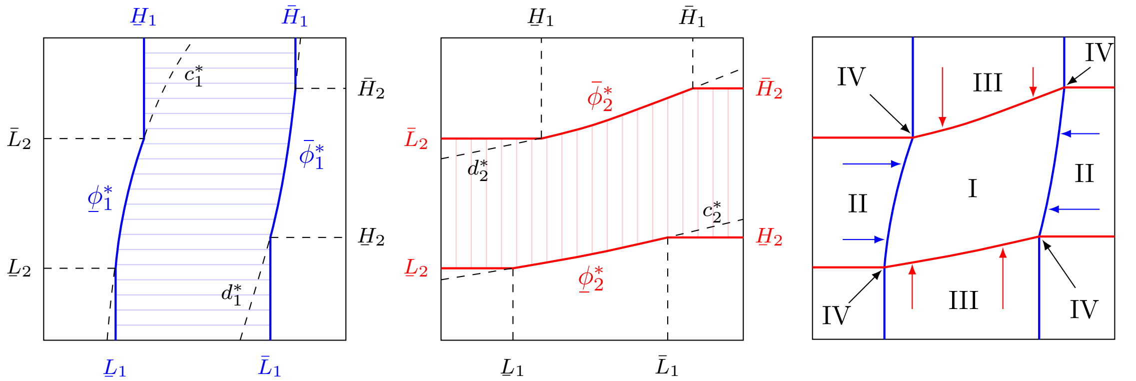
and

$$\bar{\phi}_i^*(s_{-i}) \equiv \begin{cases} \bar{L}_i, & \text{if } s_{-i} \in [0,\ \underline{H}_{-i}], \\ d_i^*(s_{-i}), & \text{if } s_{-i} \in (\underline{H}_{-i},\ \bar{H}_{-i}), \\ \bar{H}_i, & \text{if } s_{-i} \in [\bar{H}_{-i},\ 1]. \end{cases} \tag{6}$$

Then, (ϕ_1^*, ϕ_2^*) is an optimal contingent interval delegation, that is, it solves (3). Moreover, $(\phi_1^*, \phi_2^*) \in \mathcal{M}$ and if $(\phi_1, \phi_2) \in \mathcal{M}$ is also optimal, then $(\phi_1, \phi_2) = (\phi_1^*, \phi_2^*)$ over $(0, 1)$.

The construction of the optimal mechanism is illustrated by Figure 3. Panels (a) and (b) depict the resulting delegation rules $(\underline{\phi}_1^*, \bar{\phi}_1^*)$ and $(\underline{\phi}_2^*, \bar{\phi}_2^*)$ for the two agents, respectively. Take panel (a) as an example. The blue curves represent $\underline{\phi}_1^*$ and $\bar{\phi}_1^*$. As (5) defines, $\underline{\phi}_1^*$ coincides with c_1^* when $s_2 \in (\underline{L}_2, \bar{L}_2)$. It remains constant $\underline{L}_1$ when $s_2 \in [0, \underline{L}_2]$ and constant $\underline{H}_1$ when $s_2 \in [\bar{L}_2, 1]$. Analogously, $\bar{\phi}_1^*$ coincides with d_1^* when $s_2 \in (\underline{H}_2, \bar{H}_2)$. It remains constant $\bar{L}_1$ when $s_2 \in [0, \underline{H}_2]$ and constant $\bar{H}_1$ when $s_2 \in [\bar{H}_2, 1]$.

Panel (c) depicts the outcome, or equivalently the corresponding direct mechanism $(\sigma_1^{\phi_1^*}, \sigma_2^{\phi_2^*})$, under the optimal contingent interval delegation. The arrows indicate how a state is mapped to an action profile. The optimal mechanism divides the state space into four kinds of regions according to who is constrained. Region I is the



(a) Illustration of ϕ_1^* (b) Illustration of ϕ_2^* (c) Outcomes under (ϕ_1^*, ϕ_2^*)

Figure 3: Optimal mechanism

unconstrained region in the sense that both agents are able to choose their own most preferred actions. Regions II and III are the unilaterally constrained regions. In these regions, one agent (agent 2 in region II and agent 1 in region III) chooses his most preferred action, but the other agent will choose either the lower bound or the upper bound of the delegation interval for him, depending on whether his state is too low or too high. Lastly, region IV is the jointly constrained region. At each of these states, no one is able to choose his most preferred action.

The particular structure of the direct mechanism makes it group strategy-proof. That is, there is no joint misreporting that can make one agent strictly better off without hurting the other. For example, if s belongs to region II or III, one agent, say i, takes his most preferred action under truthful reporting. It is easy to see from panel (c) that there is no other decision within region I (including the boundaries) that delivers the same action for i but at the same time makes $-i$ strictly better off. If s is in region IV, the decision $\sigma^{\phi^*}(s)$ is just at one of the "vertices" of region I. It is again easy to see from the graph that there is no other decision within region I that Pareto improves upon $\sigma^{\phi^*}(s)$.[12] The following proposition summarizes the above observation.

Proposition 1 (Group strategy-proofness)**.** *The direct mechanism* $(\sigma_1^{\phi_1^*}, \sigma_2^{\phi_2^*})$ *is group strategy-proof. That is, for any states* (s_1, s_2) *and* $(\hat{s}_1, \hat{s}_2)$*, if* $v_i(\sigma_i^{\phi_i^*}(\hat{s}_i, \hat{s}_{-i}), s_i) > v_i(\sigma_i^{\phi_i^*}(s_i, s_{-i}), s_i)$*, then we must have* $v_{-i}(\sigma_{-i}^{\phi_{-i}^*}(\hat{s}_i, \hat{s}_{-i}), s_{-i}) < v_{-i}(\sigma_{-i}^{\phi_{-i}^*}(s_i, s_{-i}), s_{-i})$*.*

[12] The above argument only applies to the direct mechanism. In the indirect contingent delegation mechanism, because the range of action pairs that can arise under misreporting is strictly larger than region I, it is possible to make both agents strictly better off by joint misreporting.

3.4 A nontechnical explanation

The proof of Theorem 1 is quite involved. To explain the basic idea behind the result, we provide an informal analysis that is based on the first-order conditions. A necessary condition for (ϕ_1^*, ϕ_2^*) to be an optimal contingent interval delegation is that, for any s_{-i}, $[\underline{\phi}_i^*(s_{-i}),\ \bar{\phi}_i^*(s_{-i})]$ is an optimal single-agent delegation interval for agent i, given the other agent's behavior $\sigma_{-i}^{\phi_{-i}^*}(\,\cdot\,, s_{-i})$. Taking $i = 1$ as an example, this means that for any s_2, the pair $(\underline{\phi}_1^*(s_2),\ \bar{\phi}_1^*(s_2))$ must be a solution to

$$\max_{0\le c\le d\le 1} \int_0^1 \Big[u_0(\sigma_1(s_1; c, d), \sigma_2^{\phi_2^*}(s_1, s_2)) + u_1(\sigma_1(s_1; c, d), s_1)\Big]\,\mathrm{d}F_1(s_1). \tag{7}$$

If $\sigma_2^{\phi_2^*}(\,\cdot\,, s_2) \equiv s_2$, then this problem reduces to the unilaterally constrained delegation problem (4), and we immediately know that the solution to (7) is $(c_1^*(s_2),\ d_1^*(s_2))$ by condition U. Given ϕ_2^* from Theorem 1, this situation corresponds to the case when s_2 takes intermediate values, i.e., $\underline{H}_2 \le s_2 \le \bar{L}_2$ from panel (b) of Figure 3. For these values of s_2, Theorem 1 indeed states that $\underline{\phi}_1^*(s_2) = c_1^*(s_2)$ and $\bar{\phi}_1^*(s_2) = d_1^*(s_2)$.

However, when s_2 takes a value outside this intermediate range, $\sigma_2^{\phi_2^*}(\,\cdot\,, s_2)$ is no longer a constant. To fix ideas, consider an extremely low state s_2 so that $s_2 < \underline{L}_2$. At this state, agent 2's constrained optimal action $\sigma_2^{\phi_2^*}(s_1, s_2) = \underline{\phi}_2^*(s_1)$ for every s_1 is always higher than his unconstrained optimal action s_2. As we have mentioned in the introduction, the principal's coordination concern then would like to induce agent 1 to take higher actions. This can be done by shifting the delegation interval for agent 1 to the right of $[c_1^*(s_2),\ d_1^*(s_2)]$, as is indeed the case of $[\underline{\phi}_1^*(s_2),\ \bar{\phi}_1^*(s_2)] = [\underline{L}_1, \bar{L}_1]$ from panel (a) of Figure 3.[13]

But why is this particular interval optimal? The fundamental driving force behind this optimality is the fact that the optimal delegation interval for agent 1 is determined only by agent 2's behavior at the extreme s_1's. Intuitively, when determining the delegation interval for agent 1, the principal is considering which agent 1's extreme states to pool. From coordination point of view, this means that what matters most is agent 2's behavior at these extreme s_1's, rather than that at intermediate s_1's. Consequently, if agent 2 behaves the same at the extreme s_1's under two different contingent delegation rules, the principal's optimal action bounds for agent 1 should be the same. In particular, if agent 2's behavior is constant at the extreme s_1's, the optimal bounds for agent 1 should be the same as in the unilaterally constrained delegation problem. This is exactly the case of $\sigma_2^{\phi_2^*}(\,\cdot\,, s_2)$: $\sigma_2^{\phi_2^*}(s_1, s_2) = \underline{L}_2$ when

[13]Lemma 6 in Appendix B.1 provides a formal comparative statics result of this intuition. It also deals with potential multiplicity of the optimal intervals.

$s_1 \in [0, \underline{L}_1]$ and $\sigma_2^{\phi_2^*}(s_1, s_2) = \underline{H}_2$ when $s_1 \in [\bar{L}_1, 1]$. Therefore, the optimal lower bound for agent 1 is $c_1^*(\underline{L}_2) = \underline{L}_1$ and the optimal upper bound is $d_1^*(\underline{H}_2) = \bar{L}_1$, which is just the construction of $\underline{\phi}_1^*(s_2)$ and $\bar{\phi}_1^*(s_2)$.

To see this intuition more precisely, let us first consider the determination of $c_1^*(\underline{L}_2)$ in the principal's unilaterally constrained delegation problem. Its first order condition is

$$\int_0^{c_1^*(\underline{L}_2)} \left[\frac{\partial u_0}{\partial a_1}(c_1^*(\underline{L}_2), \underline{L}_2) + \frac{\partial u_1}{\partial a_1}(c_1^*(\underline{L}_2), s_1)\right] \mathrm{d}F_1(s_1) = 0. \tag{8}$$

To understand this first order condition, note that a change in the lower bound has two effects. First, it changes the pooling interval. Second, it changes agent 1's action over the original pooling interval. Marginally speaking, the first effect is of second order, and what really matters is the second effect. The left hand side of (8) measures this second effect. It is the change in the principal's payoff due to a marginal increase in agent 1's action over the interval $s_1 \in [0,\, c_1^*(\underline{L}_2)]$. If $c_1^*(\underline{L}_2)$ is the optimal lower bound, this payoff change must be zero. That is, (8) must hold. It is important to note that this particular payoff change only depends on how agent 2 behaves over interval $s_1 \in [0,\, c_1^*(\underline{L}_2)]$, and is independent of agent 2's behavior when $s_1 > c_1^*(\underline{L}_2)$.

Now, consider the above $s_2 < \underline{L}_2$. Although agent 2's overall behavior under ϕ_2^* at this state may differ from that when his state is $\underline{L}_2$ and he is given full discretion, they coincide when $s_1 \in [0,\, c_1^*(\underline{L}_2)]$ by construction, i.e., $\sigma_2^{\phi_2^*}(s_1, s_2) = \underline{L}_2$ for all $s_1 \in [0, c_1^*(\underline{L}_2)]$. Therefore, given $\sigma_2^{\phi_2^*}(\,\cdot\,, s_2)$, the principal should not find changing agent 1's action away from $c_1^*(\underline{L}_2)$ profitable either. Using $c_1^*(\underline{L}_2) = \underline{L}_1 = \underline{\phi}_1^*(s_2)$ by construction, we have

$$\int_0^{\underline{\phi}_1^*(s_2)} \left[\frac{\partial u_0}{\partial a_1}(\underline{\phi}_1^*(s_2), \sigma_2^{\phi_2^*}(s_1, s_2)) + \frac{\partial u_1}{\partial a_1}(\underline{\phi}_1^*(s_2), s_1)\right] \mathrm{d}F_1(s_1) = 0. \tag{9}$$

That is, $\underline{\phi}_1^*(s_2)$ satisfies one of the first order conditions for (7). Similarly, $\bar{\phi}_1^*(s_2)$ also satisfies the other first order condition, suggesting that $[\underline{\phi}_1^*(s_2),\, \bar{\phi}_1^*(s_2)]$ is indeed optimal.[14]

This derivation also explains why the boundaries of region IV (recall panel (c) of Figure 3) are all straight since it holds for any $s_2 < \underline{L}_2$. The fundamental reason is the additively separable form of the principal's payoff function. Under this form, the optimal delegation interval for agent 1 depends only on agent 2's behavior. Agent 2's state affects the optimal boundaries for agent 1 only through its effect on the behavior.

[14]See Lemma 11 in Appendix B.3 for a formal and general statement of this result.

The above explanation is based on the first order conditions, which is suggestive but far from rigorous. For instance, proving the optimality of ϕ_i^* given ϕ_{-i}^* requires checking the second order conditions. Moreover, the fact that ϕ_i^* is optimal given ϕ_{-i}^* is not enough for the optimality of joint design. To deal with these difficulties, we take a different technical approach in the formal proof, which does not explicitly rely on the first order conditions. The proof consists of two major steps. First, we indeed show that ϕ_i^* is optimal given ϕ_{-i}^*. More importantly, we show that (ϕ_1^*, ϕ_2^*) is the unique one in $\mathcal{M}$ that satisfies this property. Second, we show that among all the contingent interval delegations, there always exists an optimal one in $\mathcal{M}$. These two steps together immediately imply the optimality of (ϕ_1^*, ϕ_2^*).

Throughout the proof, complementarity of the two agents' decisions, i.e., supermodularity of u_0, guarantees that the optimal interval for one agent is monotonically increasing with respect to the other agent's behavior.[15] This property allows us to restrict attention to increasing contingent interval delegations, and it is repeatedly used in establishing both uniqueness and existence. Condition R also plays a crucial role in establishing the uniqueness.[16]

3.5 Sufficient conditions for conditions U and R

We now provide easy-to-check sufficient conditions on the payoff functions and the distributions of the states for conditions U and R to hold. These conditions are also important for (ϕ_1^*, ϕ_2^*) to be optimal among all the DSIC mechanisms.

The following lemma provides the conditions for condition U.

Lemma 3. *Condition U holds if the following conditions are satisfied:*

(U1) For all i and s_{-i}, both

$$x \mapsto \int_0^x [u_0(x, s_{-i})) + u_i(x, s_i)] \mathrm{d}F_i(s_i) + \int_x^1 [u_0(s_i, s_{-i})) + u_i(s_i, s_i)] \mathrm{d}F_i(s_i),$$

$$x \mapsto \int_0^x [u_0(s_i, s_{-i})) + u_i(s_i, s_i)] \mathrm{d}F_i(s_i) + \int_x^1 [u_0(x, s_{-i})) + u_i(x, s_i)] \mathrm{d}F_i(s_i),$$

are strictly quasi-concave.

(U2) For all i, a_i, s_i, $\frac{\partial^2 u_i}{\partial a_i \partial s_i}(a_i, s_i) > 0$.

[15] See footnote 13.

[16] Otherwise, each set of the corresponding four intersections induces a delegation rule that can potentially satisfy this property because it also satisfies the first order conditions that we discussed above.

(U3) For all i and s_{-i}, $\frac{\partial u_0}{\partial a_i}(0, s_{-i}) + \frac{\partial u_i}{\partial a_i}(0,0) \geq 0$ and $\frac{\partial u_0}{\partial a_i}(1, s_{-i}) + \frac{\partial u_i}{\partial a_i}(1,1) \leq 0$.

The first condition implies that if the principal is restricted to imposing only a floor (cap) on an agent's action in the unilaterally constrained delegation problem, the optimal floor (cap) is unique. The second condition states that if only agent i is concerned, the principal's most preferred action for agent i is strictly increasing with his state. The last condition guarantees that delegating the degenerate interval $\{0\}$ or $\{1\}$ is never a solution to the principal's unilaterally constrained delegation problem. Conditions U2 and U3 together ensure that any solution to (4) is non-degenerate, based on which condition U1 then implies that the solution is unique.

The next lemma provides the conditions for condition R on top of U.[17]

Lemma 4. *Suppose condition U is satisfied. Condition R holds if the following conditions are satisfied:*

(R1) For all i, the density function f_i is log-concave.

(R2) For all i, a and s,

$$\frac{\partial^2 u_0}{\partial a_1 \partial a_2}(a_1, a_2) \leq -\frac{\partial^2 u_0}{\partial a_i^2}(a_1, a_2). \tag{10}$$

(R3) For all i, a and s,

$$0 < \frac{\partial^2 u_i}{\partial a_i \partial s_i}(a_i, s_i) \leq -\frac{\partial^2 u_i}{\partial a_i^2}(a_i, s_i). \tag{11}$$

For example, uniform distribution, which is frequently used in the delegation literature, is log-concave.[18] Conditions R2 and R3 are about how sensitive the principal's most preferred action is with respect to the parameters. If the principal only cares about the interaction of the two actions, inequality (10) implies that her most preferred action for agent i, given that $-i$ chooses s_{-i}, is in fact not very sensitive to s_{-i}. This is because inequality (10) implies that the derivative of this action with respect to s_{-i} is bounded above by 1. Similarly, (11) implies that if the principal only cares about agent i's decision, her most preferred action given s_i is not very sensitive to s_i. These three conditions together guarantee that this insensitivity is inherited by the unilaterally constrained delegation rules. We indeed show that all the derivatives of

[17]Weaker sufficient conditions that are more difficult to check are given in Lemma 15 in the appendix.

[18]For instance, Melumad and Shibano (1991), Martimort and Semenov (2006, 2008), and Alonso et al. (2008), to name a few. See Bagnoli and Bergstrom (2005) for more examples of log-concave densities.

the unilaterally constrained delegation rules c_1^*, d_1^*, c_2^*, and d_2^* are strictly less than 1, which in turn guarantees the unique intersection of each corresponding pair in the s_1, s_2-plane. Note also that the strict inequality in (11) is just condition U2 in Lemma 3.

3.6 Optimality of contingent interval delegation

Our second main result provides conditions for the optimal contingent interval delegation (ϕ_1^*, ϕ_2^*) in Theorem 1 to be optimal among all DSIC mechanisms.

Theorem 2. *Assume conditions U1 - U3 and R1 - R2 are satisfied. If, in addition, the following conditions are satisfied, then the optimal contingent interval delegation (ϕ_1^*, ϕ_2^*) is an optimal DSIC mechanism.*

(O1) For all i, $f_i(s_i)\frac{\partial u_i}{\partial a_i}(s_i, s_i)$ is decreasing.

(O2) For all i, f_i is differentiable, and $f_i'(s_i)\frac{\partial u_i}{\partial a_i}(s_i, s_i) \geq 0$ for all s_i.

(O3) For all i, $\inf_{a_i,s_i} -\frac{\partial^2 u_i}{\partial a_i^2}(a_i, s_i) \geq \sup_{a_i,s_i} \frac{\partial^2 u_i}{\partial a_i \partial s_i}(a_i, s_i)$.

Condition O1 is one of the conditions in Proposition 5 of Alonso and Matouschek (2008), which provides sufficient conditions for interval delegation to be optimal in single-agent environments. Condition O2 requires that if only agent i is concerned, the direction of the principal's bias is the same as the direction in which f increases. Condition O3 is a strengthened version of condition R3. Conditions O1 and O2 hold simultaneously, for instance, if $\frac{\partial u_i}{\partial a_i}(s_i, s_i) = 0$ for all s_i, in which case the conflict of interests between the principal and agent i in the absence of the coordination motive essentially disappears. They also hold if f_i is the uniform distribution, in which case the monotonicity of $\frac{\partial u_i}{\partial a_i}(s_i, s_i)$ is guaranteed by condition O3.

To prove Theorem 2, we first establish a more general result, Theorem 3 in Appendix D.1. It is a verification theorem that provides sufficient conditions for a given contingent interval delegation to be optimal among all DSIC mechanisms. It is built on the main sufficiency result in Amador and Bagwell (2013), which provides sufficient conditions for a given interval delegation to be optimal in single-agent delegation problems. Theorem 3 extends their analysis to the current two-agent setting.

For Theorem 2, we show that the proposed conditions guarantee that the optimal contingent interval delegation (ϕ_1^*, ϕ_2^*) from Theorem 1 satisfies all the sufficient conditions needed in Theorem 3. Therefore, (ϕ_1^*, ϕ_2^*) is an optimal DSIC mechanism.

4 Application to Delegation in Multidivisional Organizations

4.1 Adaptation versus coordination

This application concerns multidivisional organizations where multiple decisions must be coordinated but the relevant information for decision making is dispersed among the divisions.

Consider an organization that consists of a headquarters and two divisions. The headquarters manager is the principal, while the two division managers are the agents. As we have assumed that each agent has a quadratic loss payoff function, $v_i(a_i, s_i) = \frac{1}{2}(a_i - s_i)^2$, we interpret it as that he only cares about his own adaptation loss. The principal, by contrast, cares about both the adaptation losses of the two agents and the coordination loss. Following Alonso et al. (2008), we measure the coordination loss of the two agents' actions by $-(a_1 - a_2)^2$ and assume that the principal's payoff function is[19]

$$u(a_1, a_2, s_1, s_2) \equiv -\lambda_0(a_1 - a_2)^2 - \lambda_1(a_1 - s_1)^2 - \lambda_2(a_2 - s_2)^2.$$

Here, $\lambda_0 > 0$ measures how important the coordination among the two agents is to the principal, while $\lambda_i > 0$ for $i = 1, 2$ is a parameter reflecting the importance of agent i's adaptation loss. The smaller λ_0 is or the larger λ_1 and λ_2 are, the more important the agents' adaptation loss is to the principal, and hence the less is the conflict of interest between the principal and the agents. Under this specification of the principal's payoff function, the following proposition shows that contingent interval delegation is optimal, provided that the densities of the state distributions are differentiable and log-concave.

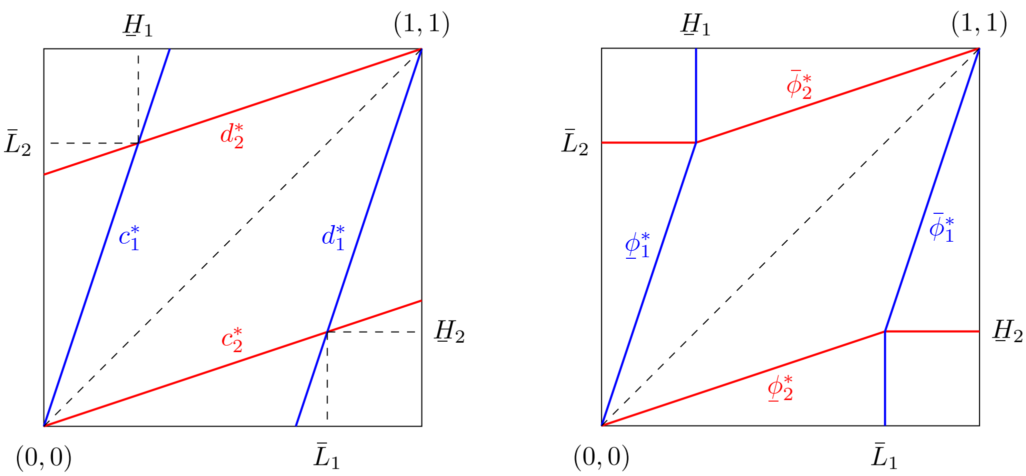
Proposition 2. *Suppose that the density functions f_1 and f_2 of the two states s_1 and s_2, respectively, are differentiable and log-concave. Then, all the sufficient conditions in Theorem 2 are satisfied. Therefore, the optimal contingent interval delegation (ϕ_1^*, ϕ_2^*) is an optimal contingent delegation. Moreover, $(\underline{L}_1, \underline{L}_2) = (0, 0)$ and $(\bar{H}_1, \bar{H}_2) = (1, 1)$, and for $i \in \{1, 2\}$, we have $0 < c_i^*(s_{-i}) < s_{-i} < d_i^*(s_{-i}) < 1$ for all $s_{-i} \in (0, 1)$.*

[19]In Alonso et al. (2008), each agent may also care about the coordination but to a lesser degree. Our model makes a simplification in this regard.

For a concrete example, consider the case where f_i is the uniform distribution over $[0,1]$. We can obtain the closed form solutions for both c_i^* and d_i^*:[20]

$$c_i^*(s_{-i}) = \frac{2\lambda_0 s_{-i}}{2\lambda_0 + \lambda_i} \quad \text{and} \quad d_i^*(s_{-i}) = \frac{2\lambda_0 s_{-i} + \lambda_i}{2\lambda_0 + \lambda_i}.$$

Panel (a) of Figure 4 illustrates these solutions for $\lambda_0 = \lambda_1 = \lambda_2$. The unique intersection of c_1^* and c_2^* is $(0,0)$ and that of d_1^* and d_2^* is $(1,1)$. Moreover, c_i^* and d_i^* always lie on different sides of the diagonal, as is claimed by Proposition 2. This is intuitive, as the principal always wants to ensure that agent i is able to choose the same action as agent $-i$, in which case perfect coordination is achieved.



(a) Unilaterally constrained delegation rules

(b) Optimal contingent interval delegation

Figure 4: Optimal mechanism for adaptation versus coordination

The corresponding optimal contingent interval delegation (ϕ_1^*, ϕ_2^*) is illustrated in panel (b) of Figure 4. Noticeably, the diagonal is completely contained in the unconstrained region. When the realized state (s_1, s_2) is on the diagonal, along which the conflict of interest between the agents and the principal vanishes, perfect adaptation and coordination are achieved simultaneously.

4.2 Comparative statics

Relative importance and optimal discretion One of the central questions in the single-agent delegation literature is how the conflict of interests between the principal

[20] Equations (C.1) and (C.2) in the online appendix provide a characterization of $c_i^*(s_{-i})$ and $d_i^*(s_{-i})$ for general log-concave density.

and the agent affects the principal's optimal mechanism. In general, less conflict of interest leads to more discretion for the agent, for example, as in Holmström (1984), Armstrong (1995), and Alonso and Matouschek (2008). In our two-agent setting, conflict of interests is measured by how important the principal thinks the agents' adaptation is relative to coordination, and it is represented by parameters λ_0, λ_1 and λ_2. The following two propositions analyze how the agents' discretion under the principal's optimal contingent delegation changes as these parameters vary. They generalize the classical single-agent result to our two-agent setting.

Proposition 3. *As coordination becomes more important to the principal, i.e., λ_0 increases, both agents will suffer from less discretion, i.e., $\underline{\phi}_i^*$ shifts upward and $\bar{\phi}_i^*$ shifts downward.*

This result should be very intuitive. In the special case $\lambda_0 = 0$, the principal does not care about coordination at all. Her delegation problem becomes two independent single-agent problems, in which the two parties' preferences are perfectly aligned. Therefore, the principal will give both agents full discretion. When $\lambda_0 > 0$, coordination between the two agents matters for the principal. It is then optimal for the principal to limit the agents' action choices for coordination. As λ_0 becomes larger, coordination becomes more important to the principal. In this case, she is willing to sacrifice more of the agents' adaptation in exchange for better coordination. Consequently, under the optimal contingent delegation, she gives both agents less discretion.

While a change in λ_0 changes the principal's overall trade-off between coordination and adaptation, the relative importance of the two agents' adaptation remains unchanged. The following proposition analyzes how this relative importance affects the agents' discretion under the optimal contingent delegation.

Proposition 4. *As agent i's adaptation becomes more important to the principal, i.e., λ_i increases, he will be granted more discretion, i.e., $\underline{\phi}_i^*$ shifts downwards and $\bar{\phi}_i^*$ shifts upwards. In contrast, agent $-i$ will suffer from less discretion, i.e., $\underline{\phi}_{-i}^*$ shifts upward and $\bar{\phi}_{-i}^*$ shifts downward.*

This first part of this proposition should also be very intuitive. When λ_i increases, the principal cares more about agent i's adaptation. Hence, it is optimal for the principal to grant more discretion to this agent for his better adaptation. As for the second part, notice that when agent i gains more discretion, he is more likely to choose his most preferred action. To avoid miscoordination, agent $-i$ then must carry more of the coordination burden. This is done by granting agent $-i$ less discretion. In the other direction, when λ_i decreases, agent i will be given less discretion but agent $-i$ will enjoy more discretion. In the limit when λ_i decreases to 0, agent $-i$ will get full discretion while agent i will lose his decision right completely: a_2 will always be set to equal agent $-i$'s decision.

As a simple corollary of Proposition 4, consider the case where state distributions of the agents are identical. If they are equally important to the principal, i.e., $\lambda_1 = \lambda_2$, the optimal delegation rules for them will be symmetric. But if one agent is more important than the other to the principal, then she will favor the more important agent by granting more discretion at the other agent's cost of receiving less discretion.

State distribution and optimal delegation rules Another aspect that affects the principal's optimal mechanism is her belief about the state distributions. For instance, if one agent's state distribution shifts to the right, how will the optimal mechanism respond? The next proposition provides the answer. It compares the optimal mechanisms when one agent's state distribution changes in the sense of the monotone likelihood ratio property (MLRP).

Proposition 5. *When one agent's state distribution increases in the sense of the MLRP, the optimal delegation rules for both agents shift upward.*

Intuitively, if agent i's state becomes more likely to be high, pooling his low states leads to smaller adaptation loss, while pooling his high states results in larger adaptation loss. Thus, it is optimal for the principal to pool more of the low states but less of the high states. That is, agent i's contingent delegation interval should move to the right. But then, it is also optimal for the principal to move agent $-i$'s contingent delegation interval to the right for coordinating with agent i's behavior.[21]

5 Conclusion

This paper studied the optimal DSIC mechanism without contingent transfers in an environment where there are two privately informed agents and the principal must decide one action for each of them. In this environment, any DSIC mechanism is equivalent to contingent delegation. We provided sufficient conditions under which

[21]We note that all the comparative statics results in this subsection can be extended to general coordination payoff function $u_0 = \lambda_0 \tilde{u}_0$, where $\tilde{u}_0$ is supermodular and $\lambda_0 \tilde{u}_0$ satisfies all the sufficient conditions for Theorem 2. The intuition remains the same.

contingent interval delegation is optimal, and solved the optimal contingent interval delegation under fairly general conditions. This optimal mechanism is determined by decomposing the two agents' joint delegation problem into single-agent ones, assuming that the other agent is free to choose his most preferred action. We also applied our results to study the delegation problem in multidivisional organizations where the two privately informed division managers only care about local adaptation but the headquarters manager also cares about coordination between the two divisions. The simple structure of the optimal mechanism enables us to analyze how conflicts of interest and state distributions affect the principal's optimal mechanism. Although we have focused on the two-agent case throughout the paper, we believe that it would not be difficult to extend our analysis to multiple agents, because the intuition of local determination can easily carry over.

One interesting question for future research is how to find the optimal Bayesian mechanism. Although the DSIC mechanism has its own conceptual advantages and makes the problem more tractable by transforming it into a contingent delegation problem, it is possible that Bayesian mechanisms can do better than DSIC mechanisms.[22] However, due to the lack of a tractable characterization of Bayesian mechanisms, it is not clear how the optimal Bayesian mechanism could be characterized. Another interesting question is whether stochastic mechanisms can improve the principal's expected payoff in our two-agent setting. In single-agent settings, it is well known that restricting attention to deterministic mechanism is in general not without loss of generality.[23] However, in a setting with quadratic preferences, Kovac and Mylovanov (2009) provide a sufficient condition for the optimal mechanism to be deterministic. It would be interesting to investigate whether a similar result holds in our setting. Since stochastic mechanisms under quadratic preferences have similar features as money burning, one possible avenue for such research is to utilize the result with money burning in Amador and Bagwell (2013).[24] We leave it for future work.

[22]The equivalence result in Gershkov et al. (2013) does not apply since v_i and u_i are not linear in a_i and we do not allow monetary transfers.

[23]See Section 8.3 in Alonso and Matouschek (2008) for a discussion.

[24]We thank an anonymous referee for pointing out this direction.

Appendix A Proofs of Lemmas 1 and 2

Proof of Lemma 1. Suppose (a_1, a_2) is a DSIC mechanism. For all i and s_{-i}, let $\tilde{D}_i(s_{-i}) \equiv \{a_i(s_i, s_{-i}) \,|\, s_i \in [0,1]\}$, and $D_i(s_{-i})$ be its closure. By DSIC, for all i, s_i and s_{-i}, $a_i(s_i, s_{-i}) \in \arg\max_{a_i' \in \tilde{D}_i(s_{-i})} v_i(a_i', s_i)$. By continuity of v_i, we know (2) holds.

Suppose (2) holds. Consider any i, s_i, s_i', and s_{-i}. Because (2) implies that $a_i(s_i', s_{-i}) \in D_i(s_{-i})$, it also implies that $v_i(a_i(s_i, s_{-i}), s_i) \geq v_i(a_i(s_i', s_{-i}), s_i)$, proving that (a_1, a_2) is a DSIC mechanism. □

Proof of Lemma 2. Continuity is standard. It comes from the maximum theorem and condition U. Monotonicity mainly comes from supermodularity of u_0. Lemma 6 in Section B.1 provides a more general statement, of which the current result is a direct corollary. See also Corollary 2. □

Appendix B Proof of Theorem 1

Throughout this section, suppose that conditions U and R hold.

B.1 One-sided optimal delegation

We begin with a generalization of unilaterally constrained delegation rules. It plays the central role throughout the whole analysis. Lemmas 5 and 6 below give its two important and useful properties.

Definition 1. Let $y : [0,1] \to [0,1]$ be a Borel measurable function. The pair (c, d) is called a *one-sided optimal delegation* for i given y, if

$$(c, d) \in \Gamma_i(y) \equiv \underset{0 \leq \tilde{c} \leq \tilde{d} \leq 1}{\arg\max} \int_0^1 \left[u_0(\sigma_i(s_i; \tilde{c}, \tilde{d}), y(s_i)) + u_i(\sigma_i(s_i; \tilde{c}, \tilde{d}), s_i) \right] \mathrm{d}F_i(s_i). \quad (12)$$

By continuity of u_0 and u_i, $\Gamma_i(y) \neq \emptyset$ for every y. Observe also that the pair $(c_i^*(s_{-i}), d_i^*(s_{-i}))$ is simply the one-sided optimal delegation for i given the constant function $y(s_i) \equiv s_{-i}$.

The following lemma points out a simple but crucial property of one-sided optimal delegations. Loosely speaking, when we consider a one-sided optimal delegation (c, d) given y, the joint optimization problem in (12) can be decomposed into two separate optimization problems, one for the lower bound c and one for the upper bound d. Most importantly, c is completely determined by the lower part of y and d is completely determined by the upper part of y.

Lemma 5 (Local determination)**.** *Suppose* $(c,d) \in \Gamma_i(y)$. *For any* x *such that* $c \leq x \leq d$, *we have*

$$c \in \underset{0 \leq \tilde{c} \leq x}{\arg\max} \int_0^x \left[u_0(\sigma_i(s_i;\tilde{c},x), y(s_i)) + u_i(\sigma_i(s_i;\tilde{c},x), s_i)\right] \mathrm{d}F_i(s_i), \tag{13}$$

$$d \in \underset{x \leq \tilde{d} \leq 1}{\arg\max} \int_x^1 \left[u_0(\sigma_i(s_i;x,\tilde{d}), y(s_i)) + u_i(\sigma_i(s_i;x,\tilde{d}), s_i)\right] \mathrm{d}F_i(s_i). \tag{14}$$

If, in addition, (c,d) *is unique, then both (13) and (14) hold with equality.*

Proof. Fix $i \in \{1,2\}$. To simplify the exposition, for every pair $0 \leq c \leq d \leq 1$ and y, let $H_i(c,d,y)$ be the function from $[0,1]$ to $\mathbb{R}$ defined as

$$H_i(c,d,y)(s_i) \equiv u_0\left(\sigma_i(s_i;c,d), y(s_i)\right) + u_i\left(\sigma_i(s_i;c,d), s_i\right), \ \forall s_i \in [0,1].$$

Hence, $\Gamma_i(y) = \arg\max_{0 \leq c \leq d \leq 1} \int_0^1 H_i(c,d,y)\mathrm{d}F_i$.

Suppose $(c,d) \in \Gamma_i(y)$ and consider any $x \in [c,d]$. En route to a contradiction, assume at least one of (13) and (14) does not hold. Pick $c' \in \arg\max_{0 \leq \tilde{c} \leq x} \int_0^x H_i(\tilde{c},x,y)\mathrm{d}F_i$ and $d' \in \arg\max_{x \leq \tilde{d} \leq 1} \int_x^1 H_i(x,\tilde{d},y)\mathrm{d}F_i$. Then, we must have

$$\int_0^x H_i(c,x,y)\mathrm{d}F_i + \int_x^1 H_i(x,d,y)\mathrm{d}F_i < \int_0^x H_i(c',x,y)\mathrm{d}F_i + \int_x^1 H_i(x,d',y)\mathrm{d}F_i. \tag{15}$$

Because $c, c' \leq x \leq d, d'$, we can easily see that the left hand side of (15) is simply $\int_0^1 H_i(c,d,y)\mathrm{d}F_i$ and the right hand side is $\int_0^1 H_i(c',d',y)\mathrm{d}F_i$. This contradicts the assumption that $(c,d) \in \Gamma_i(y)$.

From the above argument, we can also see that any pair (c',d') that satisfies $c' \in \arg\max_{0 \leq \tilde{c} \leq x} \int_0^x H_i(\tilde{c},x,y)\mathrm{d}F_i$ and $d' \in \arg\max_{x \leq \tilde{d} \leq 1} \int_{\tilde{d}}^1 H_i(x,\tilde{d},y)\mathrm{d}F_i$ must also be in $\Gamma_i(y)$. Therefore, if (c,d) is unique, we must have $(c',d') = (c,d)$. $\square$

Let Y be the set of all Borel measurable functions from $[0,1]$ to itself. We endow Y with the usual partial order $\geq$, where $y' \geq y$ if $y'(s) \geq y(s)$ for all $s \in [0,1]$. Similarly, endow $\mathbb{R}^2$ with the standard product order $\geq$, where $(c',d') \geq (c,d)$ if $c' \geq c$ and $d' \geq d$. Applying the standard results on comparative statics, we obtain the following monotonicity result.

Lemma 6 (Monotonicity)**.** *For* $i = 1,2$, *the one-sided optimal delegation correspondence* $\Gamma_i : Y \rightrightarrows [0,1]^2$ *is increasing in the strong set order.*[25] *Moreover, there exists an increasing selection of* Γ_i.

[25] That is, if $y' \geq y$, $(c,d) \in \Gamma_i(y)$ and $(c',d') \in \Gamma_i(y')$, then $(c \wedge c', d \wedge d') \in \Gamma_i(y)$ and $(c \vee c', d \vee d') \in \Gamma_i(y')$, where $c \wedge c' \equiv \min\{c,c'\}$ and $c \vee c' \equiv \max\{c,c'\}$.

Proof. We continue to use the notation $H_i(c, d, y)$ defined in the proof of Lemma 5. Let $\pi_i(c, d, y) \equiv \int_0^1 H_i(c, d, y)(s_i)\mathrm{d}F_i(s_i)$. By Theorem 2.8.3 in Topkis (1998), to show monotonicity of Γ_i, we only need to verify that (i) for every y, π_i is supermodular in (c, d), and (ii) π_i has increasing differences in $((c, d), y)$.

Fix y and consider any (c, d) and (c', d'). Without loss of generality, assume $d \leq d'$. If $c \leq c'$, we clearly have $\pi(c, d, y) + \pi(c', d', y) = \pi(c \vee c', d \vee d', y) + \pi(c \wedge c', d \wedge d', y)$. Assume $c > c'$. We thus have $c' < c \leq d \leq d'$. For any s_i, we can see

$$
\begin{aligned}
& H_i(c', d', y)(s_i) - H_i(c \wedge c', d \wedge d', y)(s_i) \\
=& H_i(c', d', y)(s_i) - H_i(c', d, y)(s_i) \\
=& \begin{cases} 0, & \text{if } s_i \leq d, \\ H_i(c, d', y)(s_i) - H_i(c, d, y)(s_i), & \text{if } s_i > d, \end{cases} \\
=& H_i(c, d', y)(s_i) - H_i(c, d, y)(s_i) \\
=& H_i(c \vee c', d \vee d', y)(s_i) - H_i(c, d, y)(s_i).
\end{aligned}
$$

Therefore, $\pi_i(c, d, y) + \pi_i(c', d', y) = \pi_i(c \vee c', d \vee d', y) + \pi_i(c \wedge c', d \wedge d', y)$, implying that π_i is supermodular (and submodular) in (c, d) for every y.

Next, consider $(c', d') \geq (c, d)$. For any y, we can easily calculate

$$
\begin{aligned}
& H_i(c', d', y)(s_i) - H_i(c, d, y)(s_i) \\
=& u_0(\sigma_i(s_i; c', d'), y(s_i)) - u_0(\sigma_i(s_i; c, d), y(s_i)) + \Delta,
\end{aligned}
$$

where $\Delta = u_i(\sigma_i(s_i; c', d'), s_i) - u_i(\sigma_i(s_i; c, d), s_i)$ is independent of y. Because $(c', d') \geq (c, d)$, we know $\sigma_i(s_i; c', d') \geq \sigma_i(s_i; c, d)$. Hence, by the supermodularity of u_0, we have, for all $y' \geq y$,

$$
H_i(c', d', y')(s_i) - H_i(c, d, y')(s_i) \geq H_i(c', d', y)(s_i) - H_i(c, d, y)(s_i), \ \forall s_i.
$$

Consequently, $\pi_i(c', d', y') - \pi_i(c, d, y') \geq \pi_i(c', d', y) - \pi_i(c, d, y)$, proving that π_i has increasing differences in $((c, d), y)$. □

Lemma 6 has two useful corollaries. Corollary 1 is used for the existence result in Section B.2, while Corollary 2 is used in the proof of uniqueness in Section B.3.

Corollary 1. *For any contingent interval delegation (ϕ_1, ϕ_2), there exists an increasing $(\phi_1', \phi_2') \in \mathcal{M}$ that yields weakly higher payoff to the principal.*

Proof. It is clear that $\sigma_2(s_2'; \underline{\phi}_2(\cdot), \bar{\phi}_2(\cdot)) \geq \sigma_2(s_2; \underline{\phi}_2(\cdot), \bar{\phi}_2(\cdot))$ whenever $s_2' > s_2$. Thus, by Lemma 6, there exists $\phi_1' = (\underline{\phi}_1', \bar{\phi}_2')$ such that (i) ϕ_1' is a one-sided optimal delegation rule for 1 given ϕ_2, and (ii) both $\underline{\phi}'_1$ and $\bar{\phi}'_1$ are increasing. Then, (ϕ'_1, ϕ_2) clearly yields an ex ante expected payoff no lower than (ϕ_1, ϕ_2) to the principal.[26] Applying the same argument, we can show that there exists $\phi'_2 = (\underline{\phi}'_2, \bar{\phi}'_2)$ such that (i) ϕ'_2 is a one-sided optimal delegation rule for agent 2 given ϕ'_1, and (ii) both $\underline{\phi}'_2$ and $\bar{\phi}'_2$ are increasing. Then (ϕ'_1, ϕ'_2) is the desired contingent interval delegation. $\square$

Corollary 2. *Suppose $y \leq (\geq)\, y'$ and $(c, d) \in \Gamma_i(y)$.*

(i) If there exists $\hat{c}$ such that every $(c', d') \in \Gamma_i(y')$ satisfies $c' = \hat{c}$, then $c \leq (\geq)\, \hat{c}$.

(ii) If there exists $\hat{d}$ such that every $(c', d') \in \Gamma_i(y')$ satisfies $d' = \hat{d}$, then $d \leq (\geq)\, \hat{d}$.

Proof. The results directly come from the definition of strong set order. $\square$

We can also naturally extend the notion of one-sided optimal delegation to mechanisms, which will give us a necessary condition for a mechanism to be optimal.

Definition 2. Consider a mechanism (ϕ_1, ϕ_2). We say ϕ_i is a *one-sided optimal delegation rule* for i given ϕ_{-i}, if, for $F_{-i}-$almost all s_{-i}, $(\underline{\phi}_i(s_{-i}), \bar{\phi}_i(s_{-i}))$ is a one-sided optimal delegation for i given $\sigma_{-i}(s_{-i}; \underline{\phi}_{-i}(\,\cdot\,), \bar{\phi}_{-i}(\,\cdot\,))$. We say (ϕ_1, ϕ_2) is a pair of *mutual one-sided optimal delegation rules* if, for both $i = 1, 2$, ϕ_i is a one-sided optimal delegation for i given ϕ_{-i}.

Being mutually one-sided optimal is a necessary condition for optimality.

Lemma 7. *If (ϕ_1, ϕ_2) is an optimal mechanism, then it is a pair of mutual one-sided optimal delegations.*

Proof. Suppose, by contradiction, that (ϕ_1, ϕ_2) is not a pair of mutual one-sided optimal delegation rules. Without loss of generality, assume that ϕ_1 is not a one-sided optimal delegation rule for 1 given ϕ_2. Consider the ϕ'_1 constructed in the proof of Corollary 1. Then it is clear that (ϕ'_1, ϕ_2) yields strictly higher ex ante expected payoff than (ϕ_1, ϕ_2) to the principal. This proves that (ϕ_1, ϕ_2) is not optimal. $\square$

Before we proceed, it is helpful to briefly discuss the main idea behind the proof of Theorem 1. Instead of directly showing that (ϕ^*_1, ϕ^*_2) performs no worse than any other contingent interval delegation, our proof takes an indirect approach. The fundamental idea of our proof is to show (i) existence — an optimal mechanism that is in $\mathcal{M}$ exists, and (ii) uniqueness — $(\bar{\phi}^*_1, \bar{\phi}^*_2)$ is the essentially unique pair of mutual one-sided optimal delegations in $\mathcal{M}$. These two results, together with Lemma 7, immediately imply the optimality of (ϕ^*_1, ϕ^*_2). The following two sections prove these two results, respectively.

[26] Monotone functions are Borel measurable.

B.2 Existence of optimal contingent interval delegation

By Corollary 1, any optimal contingent interval delegation *within* $\mathcal{M}$ is optimal for the principal. Because we can show that an optimal contingent interval delegation within $\mathcal{M}$ exists, we can obtain the desired existence result.

Lemma 8 (Existence). *Among all the contingent interval delegations, there exists an optimal one in* $\mathcal{M}$.

Proof of Lemma 8. We follow the standard line of proof that a continuous function over a compact set attains its maximum.

Consider the probability space $([0,1]^2, \mathscr{B}[0,1]^2, \mu_1 \times \mu_2)$, where $\mathscr{B}[0,1]^2$ is the Borel measurable sets over $[0,1]^2$. Each μ_i is the probability measure induced by F_i and $\mu_1 \times \mu_2$ is the product measure. Consider the following set of four dimensional random vectors over this probability space:

$$\mathcal{N} \equiv \left\{ \left(\underline{\psi}_1, \bar{\psi}_1, \underline{\psi}_2, \bar{\psi}_2\right) : [0,1]^2 \to [0,1]^4 \;\middle|\; \begin{array}{l} \underline{\psi}_1, \bar{\psi}_1 \text{ are constant in } s_1 \text{ and increasing in } s_2; \\ \underline{\psi}_2, \bar{\psi}_2 \text{ are increasing in } s_1 \text{ and constant in } s_2; \\ \forall i,\ \underline{\psi}_i(s,s) \leq \bar{\psi}_i(s,s),\ \forall s \in [0,1]. \end{array} \right\}$$

Denote a generic element in $\mathcal{N}$ by ψ. Define the distance between $\psi, \psi' \in \mathcal{N}$ as

$$\delta(\psi, \psi') \equiv \sum_{i=1}^{2} \int_0^1 \int_0^1 (|\underline{\psi}_i - \underline{\psi}'_i| + |\bar{\psi}_i - \bar{\psi}'_i|) \mathrm{d}(\mu_1 \times \mu_2).$$

As long as we regard any two random vectors ψ and ψ' as being equivalent whenever $\psi = \psi'$ a.s., δ is indeed a metric over $\mathcal{N}$.

We first show that $(\mathcal{N}, \delta)$ is compact. For this, it suffices to show that it is sequentially compact. Consider any sequence $\{\psi_n\}_n \subset \mathcal{N}$. Because of the monotonicity properties of each ψ_n, by Helly's selection theorem, there exists a pointwise convergent subsequence $\{\psi_{n_k}\}_k$ of $\{\psi_n\}_n$.[27] Let $\psi \equiv \lim_k \psi_{n_k}$. Clearly, $\psi \in \mathcal{N}$. Then, by the bounded convergence theorem, we have $\lim_k \delta(\psi_{n_k}, \psi) = 0$, proving that $(\mathcal{N}, \delta)$ is sequentially compact.

Next, we show that the mapping $\Pi : (\mathcal{N}, \delta) \to \mathbb{R}$, defined as

$$\Pi(\psi) \equiv \int_0^1 \int_0^1 \Big\{ u_0\left(\sigma_1(s_1; \underline{\psi}_1(s_1,s_2), \bar{\psi}_1(s_1,s_2)), \sigma_2(s_2; \underline{\psi}_2(s_1,s_2), \bar{\psi}_2(s_1,s_2))\right) + \sum_{i=1}^{2} u_i\left(\sigma_i(s_i; \underline{\psi}_i(s_1,s_2), \bar{\psi}_i(s_1,s_2)), s_i\right) \Big\} \mathrm{d}(\mu_1 \times \mu_2),$$

[27] See, for instance, Rudin (1976), p. 167.

is continuous. For this, we only need to show that, for any $\psi \in \mathcal{N}$ and a sequence $\{\psi_n\} \subset \mathcal{N}$ converging to ψ in δ, there is a subsequence $\{\psi_{n_k}\}_k$ such that $\Pi(\psi_{n_k}) \to \Pi(\psi)$. Because $\lim_n \delta(\psi_n, \psi) = 0$, we know that there exists a subsequence $\{\psi_{n_k}\}_k$ that converges to ψ a.s. By the bounded convergence theorem again, we know $\Pi(\psi_{n_k}) \to \Pi(\psi)$.

Finally, as Π is a continuous function over a compact set, it attains its maximum at some $\psi \in \mathcal{N}$. Define $\phi = (\underline{\phi}_1, \bar{\phi}_1, \underline{\phi}_2, \bar{\phi}_2) : [0,1] \to [0,1]^4$ as

$$\begin{aligned} \underline{\phi}_1(s_2) &\equiv \underline{\psi}_1(0, s_2),\ \ \bar{\phi}_1(s_2) \equiv \bar{\psi}_1(0, s_2),\ \ \forall s_2 \in [0,1], \\ \underline{\phi}_2(s_1) &\equiv \underline{\psi}_2(s_1, 0),\ \ \bar{\phi}_2(s_1) \equiv \bar{\psi}_2(s_1, 0),\ \ \forall s_1 \in [0,1]. \end{aligned}$$

Clearly, $\phi \in \mathcal{M}$ and is an optimal one among all the contingent interval delegations in $\mathcal{M}$. By Corollary 1, ϕ is also an optimal one among all contingent interval delegations. $\square$

B.3 Uniqueness of mutual one-sided optimal delegations in $\mathcal{M}$

Lemmas 10 and 12 below provide two necessary conditions that every pair of mutual one-sided optimal delegation rules must satisfy. Based on these two conditions, we can obtain the uniqueness.

To prove Lemma 10, we need the following lemma.

Lemma 9. *Consider $i \in \{1, 2\}$.*

(i) $c_i^*(c_{-i}^*(s_i)) > s_i$ if $s_i < \underline{L}_i$ and $c_i^*(c_{-i}^*(s_i)) < s_i$ if $s_i > \underline{L}_i$.

(ii) $d_i^*(d_{-i}^*(s_i)) > s_i$ if $s_i < \bar{H}_i$ and $d_i^*(d_{-i}^*(s_i)) < s_i$ if $s_i > \bar{H}_i$.

Proof. We show part (i). Take $i = 1$ for example. It is obvious that $(s_1, c_2^*(s_1))$ is an intersection of c_1^* and c_2^* if and only if $c_1^*(c_2^*(s_1)) = s_1$. Therefore, because of continuity of c_1^* and c_2^*, $c_1^*(c_2^*(s_1)) - s_1$ must have the same sign, either positive or negative, over $[0, \underline{L}_1)$. Because $c_1^*(c_2^*(0)) \geq 0$, we know $c_1^*(c_2^*(s_1)) - s_1$ must be positive over $[0, \underline{L}_1)$. Similarly, c_2^*, $c_1^*(c_2^*(s_1)) - s_1$ must have the same sign over $(\underline{L}_1, 1]$. Because $c_1^*(c_2^*(1)) \leq 1$, we know $c_1^*(c_2^*(s_1)) - s_1$ must be negative over $(\underline{L}_1, 1]$. $\square$

Lemma 10 (Global bounds). *Suppose $(\phi_1, \phi_2) \in \mathcal{M}$ is a pair of mutual one-sided optimal delegation rules. For $i = 1, 2$, we have $\underline{L}_i \leq \underline{\phi}_i \leq \bar{\phi}_i \leq \bar{H}_i$ over $(0, 1)$.*

Proof. For both $i = 1, 2$, we assume without loss of generality that $(\underline{\phi}_i(s_{-i}),\, \bar{\phi}_i(s_{-i}))$ is a one-sided optimal delegation for i given $\sigma_{-i}(s_{-i}; \underline{\phi}_i(\cdot), \bar{\phi}_i(\cdot))$ for $s_{-i} = 0, 1$. Otherwise, redefine $(\underline{\phi}_i(0),\, \bar{\phi}_i(0)) \equiv \lim_{s_{-i}\downarrow 0}(\underline{\phi}_i(s_{-i}),\, \bar{\phi}_i(s_{-i}))$ and $(\underline{\phi}_i(1),\, \bar{\phi}_i(1)) \equiv \lim_{s_{-i}\uparrow 1}(\underline{\phi}_i(s_{-i}),\, \bar{\phi}_i(s_{-i}))$. Because $(\underline{\phi}_i(s_{-i}),\, \bar{\phi}_i(s_{-i}))$ is a one-sided optimal delegation for i given $\sigma_{-i}(s_{-i}; \underline{\phi}_i(\cdot), \bar{\phi}_i(\cdot))$ for $F_{-i}-$almost all s_{-i} and F_{-i} has full support, such limits are also one-sided optimal delegations given the corresponding behavior.

Because $\bar{\phi}_2$ is increasing, we know $\sigma_2(1; \underline{\phi}_2(s_1), \bar{\phi}_2(s_1)) = \bar{\phi}_2(s_1) \le \bar{\phi}_2(1)$. By Corollary 2, we know

$$\bar{\phi}_1(1) \le d_1^*(\bar{\phi}_2(1)) \text{ and } \bar{\phi}_2(1) \le d_2^*(\bar{\phi}_1(1)).$$

Combining these two inequalities, we obtain

$$\bar{\phi}_1(1) \le d_1^*(d_2^*(\bar{\phi}_1(1))). \tag{16}$$

By Lemma 9, we know $\bar{\phi}_1(1) \le \bar{H}_1$, which in turn implies $\bar{\phi}_1 \le \bar{H}_1$ by monotonicity of $\bar{\phi}_1$. Similarly, we have $\bar{\phi}_2 \le \bar{H}_2$.

The other inequalities $\underline{\phi}_i \ge \underline{L}_i$ for $i = 1, 2$ can be proved analogously. □

To prove Lemma 12, we need the following lemma.

Lemma 11. *Consider $i \in \{1, 2\}$. Suppose $\underline{L}_{-i} \le \underline{s}_{-i} \le \bar{s}_{-i} \le \bar{H}_{-i}$. Let $y(s_i)$ be an increasing function that satisfies*

$$y(s_i) = \begin{cases} \underline{s}_{-i}, & \text{if } s_i \in [0,\, c_i^*(\underline{s}_{-i})], \\ \bar{s}_{-i}, & \text{if } s_i \in [d_i^*(\bar{s}_{-i}),\, 1], \end{cases} \tag{17}$$

and

$$c_i^*(y(s_i)) < s_i < d_i^*(y(s_i)),\ \forall s_i \in (c_i^*(\underline{s}_{-i}),\, d_i^*(\bar{s}_{-i})). \tag{18}$$

Then the unique one-sided optimal delegation for i given y is $(c_i^*(\underline{s}_{-i}),\, d_i^*(\bar{s}_{-i}))$.

Proof. Consider $i = 1$. We show that the optimal lower bound must be $c_1^*(\underline{s}_2)$. The proof for the upper bound is similar. Define

$$S \equiv \{ s_2 \in [\underline{s}_2, \bar{s}_2] \mid \text{every } (c, d) \in \Gamma_1(\max\{s_2, y(s_1)\}) \text{ satisfies } c = c_1^*(s_2) \}.$$

By construction of y, $\max\{\bar{s}_2, y(s_1)\} \equiv \bar{s}_2$. Because $\Gamma_1(\bar{s}_2) = \{(c_1^*(\bar{s}_2),\, d_1^*(\bar{s}_2))\}$ by condition U, we know $\bar{s}_2 \in S \neq \emptyset$. Let $\hat{s}_2 = \inf S$. For all $s_2 \in S$, we have $\hat{s}_2 \le \max\{\hat{s}_2, y(s_1)\} \le \max\{s_2, y(s_1)\}$ for all $s_1 \in [0, 1]$. Thus, by Corollary 2, any $(c, d) \in$

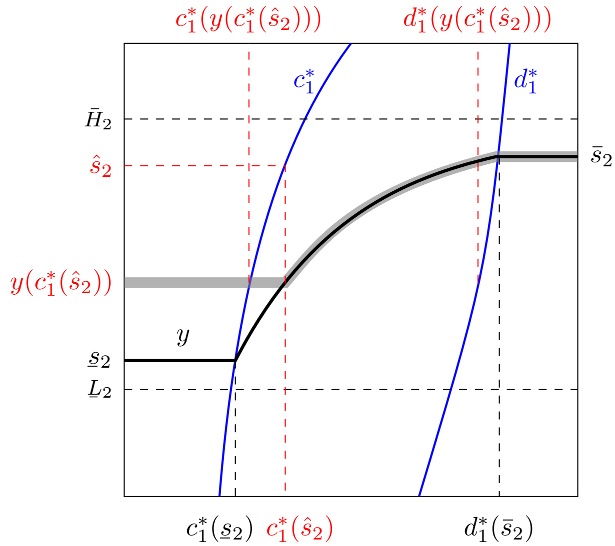


Figure 5: Proof of Lemma 11

$\Gamma_1(\max\{\hat{s}_2, y(s_1)\})$ must satisfy $c_1^*(\hat{s}_2) \leq c \leq c_1^*(s_2)$ for any $s_2 \in S$, which implies $c = c_1^*(\hat{s}_2)$ by continuity of c_1^*. Thus, $\hat{s}_2 \in S$.

The desired result will follow if we show $\hat{s}_2 = \underline{s}_2$. Suppose, by contradiction, that $\hat{s}_2 > \underline{s}_2$. In the remainder of the proof, we proceed to derive a contradiction. The analysis is divided into several small steps for clarity. In Figure 5, we carefully label the important quantities involved in the following analysis, which greatly facilitates understanding.

Step 1: $c_1^*(\underline{s}_2) < c_1^*(\hat{s}_2) < d_1^*(\bar{s}_2)$.

Because c_1^* is increasing, we know $c_1^*(\underline{s}_2) \leq c_1^*(\hat{s}_2)$. But we can not have $c_1^*(\underline{s}_2) = c_1^*(\hat{s}_2)$. To see this, note that $\underline{s}_2 \leq y(s_1) = \max\{\underline{s}_2, y(s_1)\} \leq \max\{\hat{s}_2, y(s_1)\}$ for all $s_1 \in [0,1]$. Then, for any $(c,d) \in \Gamma_1(y)$, condition U, Corollary 2 and the fact $\hat{s}_2 \in S$ together imply $c_1^*(\underline{s}_2) \leq c \leq c_1^*(\hat{s}_2)$. Consequently, equality $c_1^*(\underline{s}_2) = c_1^*(\hat{s}_2)$ would imply $\underline{s}_2 \in S$, which contradicts the definition of $\hat{s}_2$ and the assumption $\hat{s}_2 > \underline{s}_2$. Therefore, we must have $c_1^*(\underline{s}_2) < c_1^*(\hat{s}_2)$.

The other inequality comes directly from condition U and monotonicity of d_1^*: $c_1^*(\hat{s}_2) < d_1^*(\hat{s}_2) \leq d_1^*(\bar{s}_2)$.

Step 2: $c_1^*(y(c_1^*(\hat{s}_2))) < c_1^*(\hat{s}_2) < d_1^*(y(c_1^*(\hat{s}_2)))$.

This is immediate from Step 1 and the construction of y, i.e., (18).

Step 3: $\underline{s}_2 \leq y(c_1^*(\hat{s}_2)) < \hat{s}_2$.

For the first inequality, note that $\underline{s}_2 = y(c_1^*(\underline{s}_2)) \leq y(c_1^*(\hat{s}_2))$, where the equality comes from the construction of y and the inequality comes from monotonicity of both c_1^* and y. The second inequality is immediate from the first inequality in Step 2 and monotonicity of c_1^*.

Step 4: $(c, d) \in \Gamma_1(\max\{y(c_1^*(\hat{s}_2)),\, y(s_1)\})$ implies $c \leq c_1^*(\hat{s}_2) \leq d$.

By Step 3, we know $\max\{y(c_1^*(\hat{s}_2)),\, y(s_1)\} \leq \max\{\hat{s}_2,\, y(s_1)\}$. Because $\hat{s}_2 \in S$, we know $c \leq c_1^*(\hat{s}_2)$ by Corollary 2. On the other hand, because $\max\{y(c_1^*(\hat{s}_2)),\, y(s_1)\} \geq y(c_1^*(\hat{s}_2))$, we know $d \geq d_1^*(y(c_1^*(\hat{s}_2)))$ by Corollary 2 again. By Step 2, we know $d > c_1^*(\hat{s}_2)$.

Step 5: $y(c_1^*(\hat{s}_2)) \in S$.

Consider any $(c, d) \in \Gamma_1(\max\{y(c_1^*(\hat{s}_2)),\, y(s_1)\})$. Because y is increasing by construction, $\max\{y(c_1^*(\hat{s}_2)),\, y(s_1)\} = y(c_1^*(\hat{s}_2))$ for all $s_1 \in [0,\, c_1^*(\hat{s}_2)]$. By Step 4 and Lemma 5, we know

$$c \in \underset{0 \leq \tilde{c} \leq c_1^*(\hat{s}_2)}{\arg\max} \int_0^{c_1^*(\hat{s}_2)} [u_0(\sigma_1(s_1; \tilde{c}, c_1^*(\hat{s}_2)), s_1) + u_1(\sigma_1(s_1; \tilde{c}, c_1^*(\hat{s}_2)), y(c_1^*(\hat{s}_2))] \, \mathrm{d}F_1(s_1). \tag{19}$$

But by Step 2, Lemma 5 and condition U, we know that the unique solution to (19) is $c_1^*(y(c_1^*(\hat{s}_2)))$. Hence, $c = c_1^*(y(c_1^*(\hat{s}_2)))$, implying $y(c_1^*(\hat{s}_2)) \in S$.

The above Steps 3 and 5 together contradict the definition of $\hat{s}_2$. Therefore, we must have $\hat{s}_2 = \underline{s}_2$, completing the proof. $\square$

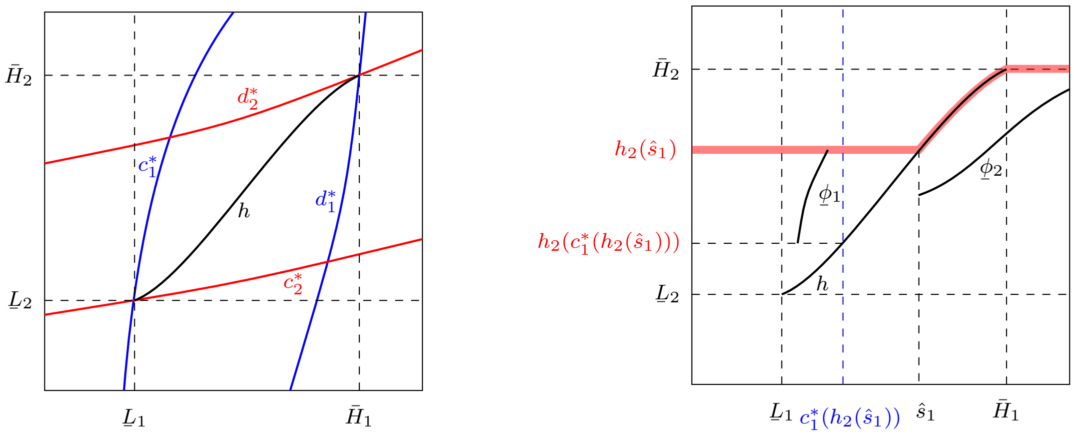
Lemma 12 (Separation)**.** *There exists a pair of mutually inverse functions h_1 and h_2 such that, for $i \in \{1, 2\}$,*

(i) $h_i : [\underline{L}_{-i}, \bar{H}_{-i}] \to [\underline{L}_i, \bar{H}_i]$ *is strictly increasing with* $h_i(\underline{L}_{-i}) = \underline{L}_i$ *and* $h_i(\bar{H}_{-i}) = \bar{H}_i$*;*

(ii) $c_i^* < h_i < d_i^*$ *over* $(\underline{L}_{-i}, \bar{H}_{-i})$*;*

and

(iii) if $(\phi_1, \phi_2) \in \mathcal{M}$ *is a pair of mutual one-sided optimal delegation rules, then* $\underline{\phi}_i \leq h_i \leq \bar{\phi}_i$ *over* $[\underline{L}_{-i}, \bar{H}_{-i}]$ *for both* $i = 1, 2$.



(a) Illustration of parts (i) and (ii)

(b) Proof of part (iii)

Figure 6: Separation property

Proof. Panel (a) of Figure 6 provides an illustration of parts (i) and (ii). It is very intuitive that we can find a strictly increasing curve (the black solid curve) that connects the two points $(\underline{L}_1, \underline{L}_2)$ and $(\bar{H}_1, \bar{H}_2)$ and that separates c_i^* and d_i^* in the sense that $c_i^* < h_i < d_i^*$. We leave its formal proof to the online appendix. Here, we show that any such h_1 and h_2 must also satisfy part (iii).

Suppose $(\underline{\phi}_1, \underline{\phi}_2)$ is a pair of mutual one-sided optimal delegation rules. Define

$$S \equiv \left\{ s_1 \in [\underline{L}_1, \bar{H}_1] \;\middle|\; \begin{array}{l} \underline{\phi}_1(s_2') \leq h_1(s_2'),\ \forall s_2' \in [h_2(s_1), \bar{H}_2], \\ \underline{\phi}_2(s_1') \leq h_2(s_1'),\ \forall s_1' \in [s_1, \bar{H}_1] \end{array} \right\}.$$

For $i = 1, 2$, we know $\underline{\phi}_i(\bar{H}_{-i}) \leq \bar{H}_i = h_i(\bar{H}_{-i})$, where the inequality comes from Lemma 10. Therefore, $\bar{H}_1 \in S \neq \emptyset$. Let $\hat{s}_1 \equiv \inf S$. It is easy to verify that $\hat{s}_1 \in S$. The desired result will follow if we show $\hat{s}_1 = \underline{L}_1$.

Suppose, by contradiction, $\hat{s}_1 > \underline{L}_1$. When $s_1 \in [\hat{s}_1, \bar{H}_1]$, we have $\underline{\phi}_2(s_1) \leq h_2(s_1)$. When $s_1 \in (\bar{H}_1, 1)$, we have $\underline{\phi}_2(s_1) \leq \bar{H}_2$ by Lemma 10. These two cases are illustrated in Figure 6. When $s_1 \in [0, \hat{s}_2)$, we know $\underline{\phi}_2(s_1) \leq \underline{\phi}_2(\hat{s}_1) \leq h_2(\hat{s}_1)$, where the first inequality comes from monotonicity of $\underline{\phi}_2$. In summary, for all s_1, we have

$$\underline{\phi}_2(s_1) \leq y(s_1) \equiv \begin{cases} h_2(\hat{s}_1), & \text{if } s_1 \in [0, \hat{s}_1), \\ h_2(s_1), & \text{if } s_1 \in [\hat{s}_1, \bar{H}_1], \\ \bar{H}_2, & \text{if } s_1 \in (\bar{H}_1, 1). \end{cases}$$

This y function is represented by the thick red curve in panel (b) of Figure 6. Consequently, for all $s_2 \in [0, h_2(\hat{s}_1)]$, we have

$$\sigma_2(s_2; \underline{\phi}_2(s_1), \bar{\phi}_2(s_1)) \leq \max\{s_2, \underline{\phi}_2(s_1)\} \leq \max\{h_2(\hat{s}_1), y(s_1)\} \leq y(s_1). \tag{20}$$

Because of parts (i) and (ii), it is easy to verify that function y satisfies conditions (17) and (18) in Lemma 11. Hence, the unique one-sided optimal delegation for 1 given y is $(c_1^*(h_2(\hat{s}_1)),\, d_1^*(\bar{H}_2))$. Because ϕ_1 is a one-sided optimal delegation rule given ϕ_2, we know that $(\underline{\phi}_1(s_2),\, \bar{\phi}_1(s_2))$ is a one-sided optimal delegation for 1 given $\sigma_2(s_2; \underline{\phi}_2(\,\cdot\,), \bar{\phi}_2(\,\cdot\,))$ for F_2-almost all $s_2 \in [0, h_2(\hat{s}_1)]$. Therefore, by (20) and Corollary 2, we know $\underline{\phi}_1(s_2) \leq c_1^*(h_2(\hat{s}_1))$ for F_2-almost all $s_2 \in [0, h_2(\hat{s}_1)]$. Because $\underline{\phi}_1$ is increasing and F_2 has full support, we actually must have $\underline{\phi}_1(s_2) \leq c_1^*(h_2(\hat{s}_1))$ for *all* $s_2 \in [0, h_2(\hat{s}_1))$. In panel (b) of Figure 6, this means that (the relevant part of) $\underline{\phi}_1$ is to the left of the vertical dashed blue line of value $c_1^*(h_2(\hat{s}_1))$. By part (ii), we know $c_1^*(h_2(\hat{s}_1)) < h_1(h_2(\hat{s}_1)) = \hat{s}_1$, where the equality comes from $h_1 = h_2^{-1}$. This in turn implies $h_2(c_1^*(h_2(\hat{s}_1))) < h_2(\hat{s}_1)$ since h_2 is strictly increasing, and

$$\underline{\phi}_1(s_2) \leq c_1^*(h_2(\hat{s}_1)) = h_1(h_2(c_1^*(h_2(\hat{s}_1)))) \leq h_1(s_2),\ \forall s_2 \in [h_2(c_1^*(h_2(\hat{s}_1))),\, h_2(\hat{s}_1)).$$

These inequalities can also be seen in panel (b) of Figure 6, as $\underline{\phi}_1$ over the interval $[h_2(c_1^*(h_2(\hat{s}_1))),\, h_2(\hat{s}_1))$ is to the left of h_1.

Initially, we know $\underline{\phi}_1(s_2) \leq h_1(s_2)$ for all $s_2 \in [h_2(\hat{s}_1),\, \bar{H}_2]$. Now, we know $\underline{\phi}_1(s_2) \leq h_1(s_2)$ for all $s_2 \in [h_2(\hat{s}_1'),\, \bar{H}_2]$, where $\hat{s}_1' \equiv c_1^*(h_2(\hat{s}_1)) < \hat{s}_1$. Similarly, using the fact that $\underline{\phi}_1(s_2) \leq h_1(s_2)$ for all $s_2 \in [h_2(\hat{s}_1),\, \bar{H}_2]$, we can also show that there exists $\hat{s}_1'' < \hat{s}_1$ such that $\underline{\phi}_2(s_1) \leq h_2(s_1)$ for all $s_1 \in [\hat{s}_1'',\, \bar{H}_1]$. This means $\max\{\hat{s}_1',\, \hat{s}_1''\} \in S$, which contradicts the definition of $\hat{s}_1$. We therefore must have $\hat{s}_2 = \underline{L}_1$. Equivalently, for both $i = 1, 2$, $\underline{\phi}_i \leq h_i$ over $[\underline{L}_{-i},\, \bar{H}_{-i}]$.

The proof of the result that $\bar{\phi}_i \geq h_i$ over $[\underline{L}_{-i},\, \bar{H}_{-i}]$ for $i = 1, 2$ is similar. □

To prove uniqueness in Lemma 14, we need the following lemma, which is analogous to Lemma 9. Its proof is omitted.

Lemma 13. *Consider $i \in \{1, 2\}$.*

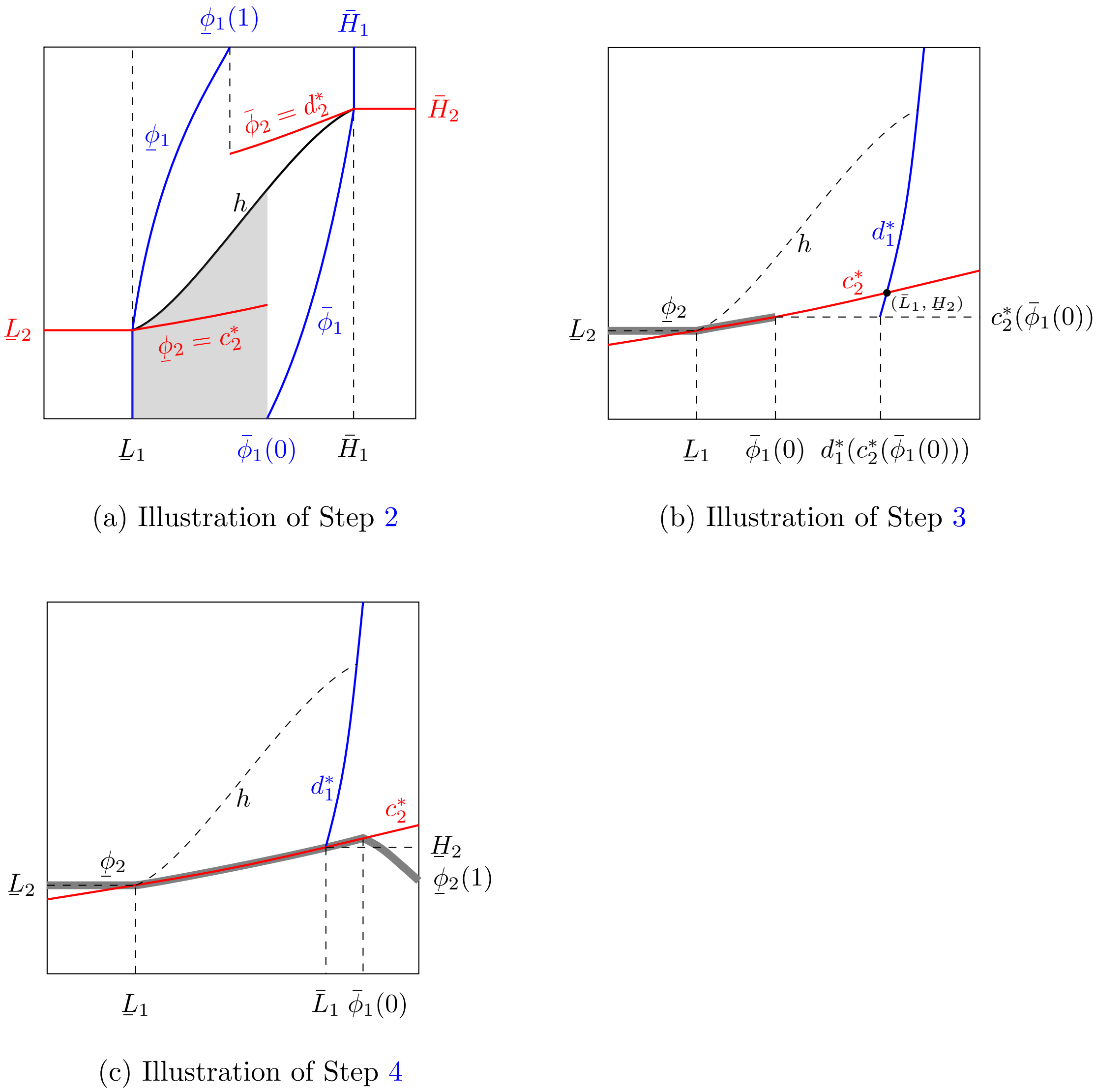
(i) $d_i^*(c_{-i}^*(s_i)) > s_i$ if $s_i < \bar{L}_i$ and $d_i^*(c_{-i}^*(s_i)) < s_i$ if $s_i > \bar{L}_i$.

(ii) $c_i^*(d_{-i}^*(s_i)) > s_i$ if $s_i < \underline{H}_i$ and $c_i^*(d_{-i}^*(s_i)) < s_i$ if $s_i > \underline{H}_i$.

We are now ready to prove uniqueness.

Lemma 14 (Uniqueness)**.** *Suppose $(\phi_1, \phi_2) \in \mathcal{M}$ is a pair of mutual one-sided optimal delegation rules. Then, we have $(\phi_1, \phi_2) = (\phi_1^*, \phi_2^*)$ over $(0, 1)$.*

Proof. Similarly as the proof of Lemma 10, assume $(\underline{\phi}_i(s_{-i}),\, \bar{\phi}_i(s_{-i}))$ is a one-sided optimal delegation for i given $\sigma_{-i}(s_{-i}; \underline{\phi}_{-i}(\,\cdot\,), \bar{\phi}_{-i}(\,\cdot\,))$ for both $s_{-i} = 0, 1$. Let h_1 and h_2 be the ones found in Lemma 12. The whole proof is divided into several small steps.



(a) Illustration of Step 2

(b) Illustration of Step 3

(c) Illustration of Step 4

Figure 7: Proof of Lemma 14

Step 1: For $i = 1, 2$, $\underline{\phi}_i(s_{-i}) = \underline{L}_i$ for all $s_{-i} \in (0,\, \underline{L}_{-i}]$, and $\bar{\phi}_i(s_{-i}) = \bar{H}_i$ for all $s_{-i} \in [\bar{H}_{-i},\, 1)$.

For $s_{-i} \in (0, \underline{L}_{-i})$, we have $\underline{L}_i \leq \underline{\phi}_i(s_{-i}) \leq \underline{\phi}_i(\underline{L}_{-i}) \leq h_i(\underline{L}_{-i}) = \underline{L}_i$, where the first inequality is from Lemma 10. The second inequality comes from monotonicity of $\underline{\phi}_i$. The third inequality comes from Lemma 12. The proof for $\bar{\phi}_i$ is similar.

Step 2: For $i = 1, 2$, $\underline{\phi}_i(s_{-i}) = c_i^*(s_{-i})$ for all $s_{-i} \in (\underline{L}_{-i},\, \bar{\phi}_{-i}(0))$, and $\bar{\phi}_i(s_{-i}) = d_i^*(s_{-i})$ for all $s_{-i} \in (\underline{\phi}_{-i}(1),\, \bar{H}_{-i})$.

Take $\underline{\phi}_2$ as an example. Consider any $s_1 \in (\underline{L}_1,\, \bar{\phi}_1(0))$ and any $s_2 \leq h_2(s_1)$. Such a pair (s_1, s_2) is a point in the shaded area in panel (a) in Figure 7. Note that

$$\underline{\phi}_1(s_2) \leq h_1(s_2) \leq h_1(h_2(s_1)) = s_1 < \bar{\phi}_1(0) \leq \bar{\phi}_1(s_2),$$

where the first inequality comes from Lemma 12. The second inequality comes from monotonicity of h_1. The last inequality comes from monotonicity of $\bar{\phi}_1$. This implies that, for all $s_1 \in (\underline{L}_1,\, \bar{\phi}_1(0))$,

$$\sigma_1(s_1; \underline{\phi}_1(s_2), \bar{\phi}_1(s_2)) = s_1,\ \forall s_2 \in (0, h_2(s_1)]. \tag{21}$$

Consider any $s_1 \in (\underline{L}_1,\, \bar{\phi}_1(0))$ such that $(\underline{\phi}_2(s_1), \bar{\phi}_2(s_1))$ is a one-sided optimal delegation given $\sigma_1(s_1; \underline{\phi}_1(\,\cdot\,), \bar{\phi}_2(\,\cdot\,))$. Because $\underline{\phi}_2(s_1) \leq h_2(s_1) \leq \bar{\phi}_2(s_1)$ by Lemma 12, Lemma 5 states that $\underline{\phi}_2(s_1)$ is completely determined by $\sigma_1(s_1; \underline{\phi}_1(\,\cdot\,), \bar{\phi}_1(\,\cdot\,))$ over $(0, h_2(s_1)]$, i.e.,

$$\underline{\phi}_2(s_1) \in \operatorname*{arg\,max}_{0 \leq \tilde{c} \leq h_2(s_1)} \int_0^{h_2(s_1)} \left[u_0(s_1, \sigma_2(s_2; \tilde{c}, h_2(s_1))) + u_2(\sigma_2(s_2; \tilde{c}, h_2(s_1)), s_2)\right] \mathrm{d}F_2(s_2). \tag{22}$$

Note that we have applied (21) in the above expression. Because $c_2^*(s_1) \leq h_2(s_2) \leq d_2^*(s_1)$ by Lemma 12, condition U and Lemma 5 then imply that the unique solution to the optimization problem in (22) is $c_2^*(s_1)$. Therefore, $\underline{\phi}_2(s_1) = c_2^*(s_1)$.

Because $(\underline{\phi}_2(s_1), \bar{\phi}_2(s_1))$ is a one-sided optimal delegation given $\sigma_1(s_1; \underline{\phi}_1(\,\cdot\,), \bar{\phi}_2(\,\cdot\,))$ for F_1-almost all $s_1 \in (\underline{L}_1,\, \bar{\phi}_1(0))$, we know from the above analysis that $\underline{\phi}_2(s_1) = c_2^*(s_1)$ for F_1-almost all $s_1 \in (\underline{L}_1,\, \bar{\phi}_1(0))$. Because $\underline{\phi}_2$ is increasing, c_2^* is continuous and F_1 has full support, we have $\underline{\phi}_2(s_1) = c_2^*(s_1)$ for all $s_1 \in (\underline{L}_1,\, \bar{\phi}_1(0))$.

Step 3: For $i = 1, 2$, we must have $\bar{\phi}_i(0) \geq \bar{L}_i$ and $\underline{\phi}_i(1) \leq \underline{H}_i$.

We take $\bar{\phi}_1(0) \geq \bar{L}_1$ as an example. Other inequalities are similar. Suppose, by contradiction, that $\bar{\phi}_1(0) < \bar{L}_1$. This situation is illustrated in panel (b) of Figure 7.

The thick gray curve is $\underline{\phi}_2$. By Steps 1 and 2, we know $\underline{\phi}_2$ is constant $\underline{L}_2$ over $(0, \underline{L}_1]$ and coincides with c_2^* over $(\underline{L}_1,\, \bar{\phi}_1(0))$. Because $\underline{\phi}_2$ is increasing, for all $s_1 \in [\bar{\phi}_1(0),\, 1]$, we know

$$\underline{\phi}_2(s_1) \geq \lim_{s_1' \uparrow \bar{\phi}_1(0)} \underline{\phi}_2(s_1') = \lim_{s_1' \uparrow \bar{\phi}_1(0)} c_2^*(s_1') = c_2^*(\bar{\phi}_1(0)).$$

Therefore, we have

$$\underline{\phi}_2(s_1) \geq y(s_1) \equiv \begin{cases} \underline{L}_2, & \text{if } s_1 \in (0, \underline{L}_1], \\ c_2^*(s_1), & \text{if } s_1 \in (\underline{L}_1,\, \bar{\phi}_1(0)), \\ c_2^*(\bar{\phi}_1(0)), & \text{if } s_1 \in (\bar{\phi}_1(0),\, 1]. \end{cases}$$

This in turn implies that

$$\sigma_2(0; \underline{\phi}_2(s_1), \bar{\phi}_2(s_1)) = \underline{\phi}_2(s_1) \geq y(s_1),\ \forall s_1 \in [0, 1]. \tag{23}$$

It is easy to check that this y function satisfies conditions (17) and (18) in Lemma 11. Hence, the unique one-sided optimal delegation rule for agent 1 given y is $(\underline{L}_1,\, d_1^*(c_2^*(\bar{\phi}_1(0))))$. Because $(\underline{\phi}_1(0),\, \bar{\phi}_1(0))$ is a one-sided optimal delegation given $\sigma_2(0; \underline{\phi}_2(\,\cdot\,), \bar{\phi}_2(\,\cdot\,))$, we know $\bar{\phi}_1(0) \geq d_1^*(c_2^*(\bar{\phi}_1(0)))$ by inequality (23) and Corollary 2. By Lemma 13, we know $\bar{\phi}_1(0) \geq \bar{L}_1$, contradicting our assumption that $\bar{\phi}_1(0) < \bar{L}_1$. Therefore, we must have $\bar{\phi}_1(0) \geq \bar{L}_1$.

Step 4: For $i = 1, 2$, we must have $\bar{\phi}_i(0) = \bar{L}_i$ and $\underline{\phi}_i(1) = \underline{H}_i$.

Panel (c) of Figure 7 illustrates what would happen if $\bar{\phi}_1(0) > \bar{L}_1$ when c_2^* is strictly increasing. Again, the thick gray curve represents $\underline{\phi}_2$. By Step 2, we know $\underline{\phi}_2$ will go above $\underline{H}_2$ over $(\bar{L}_1,\, \bar{\phi}_1(0))$ as c_2^* does. But Step 3 claims that $\underline{\phi}_2(1) \leq \underline{H}_2$. Therefore, this is impossible because $\underline{\phi}_2$ is increasing.

More formally, note that the following chain of inequalities must hold

$$\bar{\phi}_1(0) \leq \bar{\phi}_1(\underline{\phi}_2(1)) \leq d_1^*(\underline{H}_2) = \bar{L}_1 \leq \bar{\phi}_1(0),$$

where the first inequality comes from monotonicity of $\bar{\phi}_1$. The second inequality comes from Steps 2 and 3. The last one comes from Step 3. Therefore, we have $\bar{\phi}_1(0) = \bar{L}_1$. The other equalities can be similarly proved.

Step 5: For $i = 1, 2$, $\underline{\phi}_i(s_{-i}) = \underline{H}_i$ for all $s_{-i} \in [\bar{L}_{-i},\, 1]$ and $\bar{\phi}_i(s_{-i}) = \bar{L}_i$ for all $s_{-i} \in [0,\, \underline{H}_{-i}]$.

This is obvious now. For example, we have

$$\underline{H}_2 = c_2^*(\bar{L}_1) \leq \underline{\phi}_2(\bar{L}_1) \leq \underline{\phi}_2(1) = \underline{H}_2,$$

where the first inequality comes from Steps 2 and 4. The second inequality comes from monotonicity of $\underline{\phi}_2$. Therefore, we have $\underline{\phi}_2(\bar{L}_1) = \underline{\phi}_2(1) = \underline{H}_2$. By monotonicity of $\underline{\phi}_2$ again, we know $\underline{\phi}_2(s_1) \equiv \underline{H}_2$ for $s_1 \in [\bar{L}_1, 1]$.

Combining Steps 1, 2 and 5 yields the desired result. $\square$

Proof of Theorem 1. Lemmas 7, 8, and 14 together prove Theorem 1. $\square$

Appendix C Proofs of Lemmas 3 and 4

C.1 Proof of Lemma 3

Proof. For notational simplicity, let

$$\underline{g}_i(x, s_{-i}) \equiv \int_0^x [u_0(x, s_{-i})) + u_i(x, s_i)]\mathrm{d}F_i(s_i) + \int_x^1 [u_0(s_i, s_{-i})) + u_i(s_i, s_i)]\mathrm{d}F_i(s_i),$$
$$\bar{g}_i(x, s_{-i}) \equiv \int_0^x [u_0(s_i, s_{-i})) + u_i(s_i, s_i)]\mathrm{d}F_i(s_i) + \int_x^1 [u_0(x, s_{-i})) + u_i(x, s_i)]\mathrm{d}F_i(s_i).$$

Fix s_{-i}. It is easy to note that (4) can be equivalently written as

$$\max_{0 \le c \le d \le 1} \underline{g}_i(c, s_{-i}) + \bar{g}_i(d, s_{-i}). \tag{24}$$

We proceed to show that this optimization problem has a unique solution, which is non-degenerate.

Consider any solution $(\hat{c}, \hat{d})$ to (24). We first claim that

$$\frac{\partial \underline{g}_i}{\partial x}(\hat{c}, s_{-i}) = \int_0^{\hat{c}} \left(\frac{\partial u_0}{\partial a_i}(\hat{c}, s_{-i}) + \frac{\partial u_i}{\partial a_i}(\hat{c}, s_i) \right) \mathrm{d}F_i(s_i) \ge 0, \tag{25}$$
$$\frac{\partial \bar{g}}{\partial x}(\hat{d}, s_{-i}) = \int_{\hat{d}}^1 \left(\frac{\partial u_0}{\partial a_i}(\hat{d}, s_{-i}) + \frac{\partial u_i}{\partial a_i}(\hat{d}, s_i) \right) \mathrm{d}F_i(s_i) \le 0. \tag{26}$$

For instance, if (25) is violated, i.e., $\frac{\partial \underline{g}_i}{\partial x}(\hat{c}, s_{-i}) < 0$, we know $\hat{c} > 0$ because $\frac{\partial \underline{g}_i}{\partial x}(0, s_{-i}) = 0$. Then, there exists $c \in [0, \hat{c})$ such that $\underline{g}_i(c, s_{-i}) > \underline{g}_i(\hat{c}, s_{-i})$. This, in turn, implies that $\underline{g}_i(c, s_{-i}) + \bar{g}_i(\hat{d}, s_{-i}) > \underline{g}_i(\hat{c}, s_{-i}) + \bar{g}_i(\hat{d}, s_{-i})$. Because $(c, \hat{d})$ is also feasible to (24), we know $(\hat{c}, \hat{d})$ is not a solution, which is a contradiction. Therefore, (25) must hold. Using a similar argument, we can see that (26) must hold too.

Next, we claim that $\hat{c} < \hat{d}$. Suppose, by contradiction, $\hat{c} = \hat{d} \equiv \hat{x}$. (U2) implies that, for all x, $\frac{\partial u_0}{\partial a_i}(x, s_{-i}) + \frac{\partial u_i}{\partial a_i}(x, s_i)$ is strictly increasing in s_i. Hence, (U3) then implies that $\frac{\partial \underline{g}_i}{\partial x}(1, s_{-i}) < 0$ and $\frac{\partial \bar{g}_i}{\partial x}(0, s_{-i}) > 0$. By (25) and (26), we know $\hat{c} < 1$ and

$\hat{d} > 0$, implying that $\hat{x} \in (0,1)$. Then, (25) and (U2) together imply $\frac{\partial u_0}{\partial a_i}(\hat{x}, s_{-i}) + \frac{\partial u_i}{\partial a_i}(\hat{x}, \hat{x}) > 0$. Likewise, (26) and (U2) together imply $\frac{\partial u_0}{\partial a_i}(\hat{x}, s_{-i}) + \frac{\partial u_i}{\partial a_i}(\hat{x}, \hat{x}) < 0$, a contradiction.

Finally, we show that $(\hat{c}, \hat{d})$ is the unique solution to (24) (and hence to (4)). Because $\underline{g}_i(\,\cdot\,, s_{-i})$ is strictly quasi-concave by (U1), $\max_{0\le c\le 1} \underline{g}_i(c, s_{-i})$ has a unique solution. Denote this solution by $\tilde{c}$. If $\tilde{c} < \hat{c}$, we know $(\tilde{c}, \hat{d})$ is feasible to (24), and $\underline{g}_i(\tilde{c}, s_{-i}) + \bar{g}_i(\hat{d}, s_{-i}) > \underline{g}_i(\hat{c}, s_{-i}) + \bar{g}_i(\hat{d}, s_{-i})$, contradicting the optimality of $(\hat{c}, \hat{d})$. If $\tilde{c} > \hat{c}$, we know $\underline{g}_i(\,\cdot\,, s_{-i})$ is strictly increasing over $[\hat{c}, \tilde{c}]$ by strict quasi-concavity. Pick $c \in (\hat{c}, \min\{\tilde{c}, \hat{d}\})$. Then, $(c, \hat{d})$ is feasible to (24), and $\underline{g}_i(c, s_{-i}) + \bar{g}_i(\hat{d}, s_{-i}) > \underline{g}_i(\hat{c}, s_{-i}) + \bar{g}_i(\hat{d}, s_{-i})$, contradicting the optimality of $(\hat{c}, \hat{d})$ again. Therefore, we must have $\hat{c} = \tilde{c}$. Similarly, using the strict quasi-concavity of $\bar{g}_i(\,\cdot\,, s_{-i})$, we can show that $\hat{d}$ is the unique solution to $\max_{0\le d\le 1} \bar{g}_i(x, s_{-i})$, completing the proof. $\square$

C.2 Proof of Lemma 4

To prove Lemma 4, we first prove Lemmas 15 and 16 below. Lemma 15 itself can be considered as weaker sufficient conditions for condition R.

Lemma 15. *Suppose condition U is satisfied. If the following conditions are satisfied, condition R holds: for all i, $a_i \in (0,1)$ and s_{-i},*

$$\begin{aligned} &\left(\frac{\partial^2 u_0}{\partial a_i \partial a_{-i}}(a_i, s_{-i}) + \frac{\partial^2 u_0}{\partial a_i^2}(a_i, s_{-i})\right) F_i(a_i) \\ &< \int_0^{a_i} \left(-\frac{f_i(a_i)}{F_i(a_i)} \frac{F_i(s_i)}{f_i(s_i)} \frac{\partial^2 u_i}{\partial a_i \partial s_i}(a_i, s_i) - \frac{\partial^2 u_i}{\partial a_i^2}(a_i, s_i)\right) \mathrm{d}F_i(s_i), \qquad (27) \\ &\left(\frac{\partial^2 u_0}{\partial a_i \partial a_{-i}}(a_i, s_{-i}) + \frac{\partial^2 u_0}{\partial a_i^2}(a_i, s_{-i})\right) (1 - F_i(a_i)) \\ &< \int_{a_i}^1 \left(-\frac{f_i(a_i)}{1 - F_i(a_i)} \frac{1 - F_i(s_i)}{f_i(s_i)} \frac{\partial^2 u_i}{\partial a_i \partial s_i}(a_i, s_i) - \frac{\partial^2 u_i}{\partial a_i^2}(a_i, s_i)\right) \mathrm{d}F_i(s_i). \qquad (28) \end{aligned}$$

Proof. We first claim that, for all i, c_i^* is differentiable at s_{-i} such that $c_i^*(s_{-i}) > 0$, and $\frac{\mathrm{d}c_i^*(s_{-i})}{\mathrm{d}s_{-i}} < 1$. Because $c_i^*(s_{-i}) < d_i^*(s_{-i})$ by condition U, from the first two paragraphs of the proof of Lemma 3, we know the first order condition is satisfied:

$$\frac{\partial \underline{g}_i}{\partial x}(c_i^*, s_{-i}) = \int_0^{c_i^*} \left(\frac{\partial u_0}{\partial a_i}(c_i^*, s_{-i}) + \frac{\partial u_i}{\partial a_i}(c_i^*, s_i)\right) \mathrm{d}F_i(s_i) = 0, \qquad (29)$$

where we write c_i^* instead of $c_i^*(s_{-i})$ for short. It is easy to calculate

$$\begin{aligned}
&-\frac{\partial^2 \underline{g}_i}{\partial x^2}(c_i^*, s_{-i}) \\
=&-\Big(\frac{\partial u_0}{\partial a_i}(c_i^*, s_{-i}) + \frac{\partial u_i}{\partial a_i}(c_i^*, c_i^*)\Big) f_i(c_i^*) - \int_0^{c_i^*} \bigg(\frac{\partial^2 u_0}{\partial a_i^2}(c_i^*, s_{-i}) + \frac{\partial^2 u_i}{\partial a_i^2}(c_i^*, s_i)\bigg) \mathrm{d}F_i(s_i) \\
=&\int_0^{c_i^*} \bigg(\frac{f_i(c_i^*)}{F_i(c_i^*)}\Big(\frac{\partial u_i}{\partial a_i}(c_i^*, s_i) - \frac{\partial u_i}{\partial a_i}(c_i^*, c_i^*)\Big) - \frac{\partial^2 u_0}{\partial a_i^2}(c_i^*, s_{-i}) - \frac{\partial^2 u_i}{\partial a_i^2}(c_i^*, s_i)\bigg) \mathrm{d}F_i(s_i) \\
=&\int_0^{c_i^*} \bigg(-\frac{f_i(c_i^*)}{F_i(c_i^*)}\frac{F_i(s_i)}{f_i(s_i)}\frac{\partial^2 u_i}{\partial a_i \partial s_i}(c_i^*, s_i) - \frac{\partial^2 u_i}{\partial a_i^2}(c_i^*, s_i)\bigg) \mathrm{d}F_i(s_i) - \frac{\partial^2 u_0}{\partial a_i^2}(c_i^*, s_{-i}) F_i(c_i^*) \\
>&\frac{\partial^2 u_0}{\partial a_i \partial a_{-i}}(c_i^*, s_{-i}) F_i(c_i^*) \geq 0, \qquad (30)
\end{aligned}$$

where the second equality comes from the first order condition (29). The third equality comes from $\int_0^{c_i^*}(\frac{\partial u_i}{\partial a_i}(c_i^*, s_i) - \frac{\partial u_i}{\partial a_i}(c_i^*, c_i^*))\mathrm{d}F_i(s_i) = -\int_0^{c_i^*}\int_{s_i}^{c_i^*} \frac{\partial^2 u_i}{\partial a_i \partial s_i}(c_i^*, \tilde{s}_i)\mathrm{d}\tilde{s}_i \mathrm{d}F_i(s_i) = -\int_0^{c_i^*} \frac{\partial^2 u_i}{\partial a_i \partial s_i}(c_i^*, \tilde{s}_i)\Big(\int_0^{\tilde{s}_i} \mathrm{d}F_i(s_i)\Big)\mathrm{d}\tilde{s}_i = -\int_0^{c_i^*} \frac{F_i(s_i)}{f_i(s_i)}\frac{\partial^2 u_i}{\partial a_i \partial s_i}(c_i^*, s_i)\mathrm{d}F_i(s_i)$. The first inequality comes from (27). The last inequality comes from $\frac{\partial^2 u_0}{\partial a_1 \partial a_2} \geq 0$. Therefore, by the implicit function theorem, we know $c_i^*(s_{-i})$ is differentiable when it is positive, and

$$\frac{\mathrm{d}c_i^*(s_{-i})}{\mathrm{d}s_{-i}} = \frac{\frac{\partial^2 \underline{g}_i}{\partial x \partial s_{-i}}(c_i^*, s_{-i})}{-\frac{\partial^2 \underline{g}_i}{\partial x^2}(c_i^*, s_{-i})} = \frac{\frac{\partial^2 u_0}{\partial a_i \partial a_{-i}}(c_i^*, s_{-i}) F_i(c_i^*)}{-\frac{\partial^2 \underline{g}_i}{\partial x^2}(c_i^*, s_{-i})} < 1,$$

where the inequality comes from (30).

Similarly, using the first order condition $\frac{\partial \bar{g}_i}{\partial x}(d_i^*, s_{-i}) = 0$ and (28), we can show that $d_i^*(s_{-i})$ is also differentiable at s_{-i} such that $d_i^*(s_{-i}) < 1$, and $\frac{\mathrm{d}d_i^*(s_{-i})}{\mathrm{d}s_{-i}} < 1$.

Next, we claim that, for all i, $c_i^*(s'_{-i}) - c_i^*(s_{-i}) < s'_{-i} - s_{-i}$ and $d_i^*(s'_{-i}) - d_i^*(s_{-i}) < s'_{-i} - s_{-i}$ for all $s'_{-i} > s_{-i}$. Take c_i^* as an example. Because $c_i^* \geq 0$ is increasing by Lemma 2, it takes one of the following three forms: (i) $c_i^* \equiv 0$ over $[0,1]$; (ii) $c_i^* > 0$ over $[0,1]$; and (iii) there exists $\hat{s} \in [0,1)$ such that $c_i^* = 0$ over $[0, \hat{s}]$ and $c_i^* > 0$ over $(\hat{s}, 1]$. From the above analysis, we can see that, in all cases, c_i^* is absolutely continuous and its derivative is strictly less than 1. Therefore $c_i^*(s'_{-i}) - c_i^*(s_{-i}) = \int_{s_{-i}}^{s'_{-i}} \frac{\mathrm{d}c_i^*(\tilde{s}_{-i})}{\mathrm{d}s_{-i}}\mathrm{d}\tilde{s}_{-i} < s'_{-i} - s_{-i}$ for all $s'_{-i} > s_{-i}$. The argument for d_i^* is similar.

Finally, for any $s'_i > s_i$, we have $c_i^*(c_{-i}^*(s'_i)) - c_i^*(c_{-i}^*(s_i)) \leq c_{-i}^*(s'_i) - c_{-i}^*(s_i) < s'_i - s_i$ and $c_i^*(d_{-i}^*(s'_i)) - c_i^*(d_{-i}^*(s_i)) \leq d_{-i}^*(s'_i) - d_{-i}^*(s_i) < s'_i - s_i$. Therefore, each of $c_1^* \circ c_2^*$, $c_1^* \circ d_2^*$, $d_1^* \circ c_2^*$, and $d_1^* \circ d_2^*$ has a unique fixed point. □

The following lemma summarizes some useful properties of the distribution functions derived from log-concavity of the density function. Some of these properties are used in the proof of Lemma 4. Some are used later in the proof of Theorem 2.

Lemma 16. *If f_i is log-concave, then F_i, $1-F_i$, $\int_0^{s_i} F_i(x)\mathrm{d}x$, and $\int_{s_i}^1 (1-F_i(x))\mathrm{d}x$ are all log-concave. Consequently, $\frac{f_i'}{f_i}$, $\frac{f_i}{F_i}$, $\frac{-f_i}{1-F_i}$, $\frac{F_i(s_i)}{\int_0^{s_i} F_i(x)\mathrm{d}x}$, and $\frac{-(1-F_i(s_i))}{\int_{s_i}^1 (1-F_i(x))\mathrm{d}x}$ are all decreasing. Moreover,*

(i) $\lim_{s_i \downarrow 0} \frac{f_i(s_i)}{F_i(s_i)} = \lim_{s_i \uparrow 1} \frac{f_i(s_i)}{1-F_i(s_i)} = +\infty$.

(ii) $\frac{f_i'(s_i)}{f_i(s_i)} \frac{\int_0^{s_i} F_i(\tilde{s}_i)\mathrm{d}\tilde{s}_i}{F_i(s_i)}$, $\frac{-f_i'(s_i)}{f_i(s_i)} \frac{\int_{s_i}^1 (1-F_i(\tilde{s}_i))\mathrm{d}\tilde{s}_i}{1-F_i(s_i)}$, $\frac{f_i(s_i)\int_0^{s_i} F_i(x)\mathrm{d}x}{F_i^2(s_i)}$, *and* $\frac{f_i(s_i)\int_{s_i}^1 (1-F_i(x))\mathrm{d}x}{(1-F_i(s_i))^2}$ *are all bounded above by* 1.

Proof. Because f_i is log-concave, it is well-known that all these derived functions are log-concave.[28] Monotonicity of the first order derivatives of the logarithm of these functions follow directly.

Consider part (i). Because $\lim_{s_i \downarrow 0} \int_{s_i}^{\frac{1}{2}} \frac{f_i(\tilde{s}_i)}{F_i(\tilde{s}_i)}\mathrm{d}\tilde{s}_i = \log F_i(\frac{1}{2}) - \lim_{s_i \downarrow 0} \log F_i(s_i) = +\infty$, monotonicity of $\frac{f_i}{F_i}$ implies $\lim_{s_i \downarrow 0} \frac{f_i(s_i)}{F_i(s_i)} = +\infty$. The other limit is analogous.

Consider part (ii). It is easy to observe that

$$\operatorname{sign}\left(1 - \frac{f_i'(s_i)}{f_i(s_i)} \frac{\int_0^{s_i} F_i(\tilde{s}_i)\mathrm{d}\tilde{s}_i}{F_i(s_i)}\right) = \operatorname{sign}\left(\frac{\int_0^{s_i} F_i(x)\mathrm{d}x}{f_i(s_i)}\right)' = \operatorname{sign}\left(\frac{F_i(s_i)}{f_i(s_i)} \frac{\int_0^{s_i} F_i(x)\mathrm{d}x}{F_i(s_i)}\right)' \geq 0.$$

The other inequalities can be similarly proved. □

Proof of Lemma 4. By Lemma 15, we only need to verify that conditions R1 - R3 imply (27) and (28). Inequalities (10) and (11) imply that, for all i, a_i and s_{-i},

$$\left(\frac{\partial^2 u_0}{\partial a_i \partial a_{-i}}(a_i, s_{-i}) + \frac{\partial^2 u_0}{\partial a_i^2}(a_i, s_{-i})\right) F_i(a_i) \leq \int_0^{a_i} \left(-\frac{\partial^2 u_i}{\partial a_i \partial s_i}(a_i, s_i) - \frac{\partial^2 u_i}{\partial a_i^2}(a_i, s_i)\right) \mathrm{d}F_i(s_i).$$

For all $a_i \in (0,1)$, the first inequality in (11) and Lemma 16 together imply that

$$-\frac{\partial^2 u_i}{\partial a_i \partial s_i}(a_i, s_i) \leq -\frac{f_i(a_i)}{F_i(a_i)} \frac{F_i(s_i)}{f_i(s_i)} \frac{\partial^2 u_i}{\partial a_i \partial s_i}(a_i, s_i), \ \forall s_i < a_i,$$

with strict inequality when s_i is sufficiently small. Thus,

$$\int_0^{a_i} -\frac{\partial^2 u_i}{\partial a_i \partial s_i}(a_i, s_i)\mathrm{d}F_i(s_i) < \int_0^{a_i} -\frac{f_i(a_i)}{F_i(a_i)} \frac{F_i(s_i)}{f_i(s_i)} \frac{\partial^2 u_i}{\partial a_i \partial s_i}(a_i, s_i)\mathrm{d}F_i(s_i).$$

Therefore, (27) holds. We can similarly show that (28) holds too. □

[28] See, for instance, An (1998) and Bagnoli and Bergstrom (2005).

Appendix D Proof of Theorem 2

D.1 Optimality of contingent interval delegation

Theorem 2 in the main text provides sufficient conditions for the particular optimal contingent interval delegation (ϕ_1^*, ϕ_2^*) from Theorem 1 to be optimal among all DSIC mechanisms. As we have explained in the main text, it is based on a more general result that provides conditions for a given contingent interval delegation to be optimal. Because this result has its own interest, we state it below as a theorem. It generalizes the main sufficiency result in Amador and Bagwell (2013).

Theorem 3. *Consider a contingent interval delegation (ϕ_1, ϕ_2). For each i, define*

$$w_i(a_i, s_i, s_{-i}) \equiv u_i(a_i, s_i) + u_0(a_i, \sigma_{-i}^{\phi_{-i}}(s_i, s_{-i})),$$
$$\kappa_i \equiv \inf_{a_i, s_i \in [0,1]} -\frac{\partial^2 u_i}{\partial a_i^2}(a_i, s_i). \tag{31}$$

If the conditions C1, C2, C2′, C3 and C3′ are satisfied, then $(\sigma_1^{\phi_1}, \sigma_2^{\phi_2})$ is an optimal DSIC mechanism.

(C1) For any $s_{-i} \in [0,1]$,

$$\kappa_i F_i(s_i) - f_i(s_i)\frac{\partial w_i}{\partial a_i}(s_i, s_i, s_{-i})$$

is increasing in s_i for $s_i \in [\underline{\phi}_i(s_{-i}), \bar{\phi}_i(s_{-i})]$.

(C2) If $\underline{\phi}_i(s_{-i}) > 0$,

$$(s_i - \underline{\phi}_i(s_{-i}))\kappa_i \leq \int_0^{s_i} \frac{\partial w_i}{\partial a_i}(\underline{\phi}_i(s_{-i}), \tilde{s}_i, s_{-i})\frac{f_i(\tilde{s}_i)}{F_i(s_i)}\mathrm{d}\tilde{s}_i, \quad \forall s_i \in [0, \underline{\phi}_i(s_{-i})],$$

with equality at $\underline{\phi}_i(s_{-i})$.

(C2′) If $\underline{\phi}_i(s_{-i}) = 0$, $\frac{\partial w_i}{\partial a_i}(0, 0, s_{-i}) \leq 0$.

(C3) If $\bar{\phi}_i(s_{-i}) < 1$,

$$(s_i - \bar{\phi}_i(s_{-i}))\kappa_i \geq \int_{s_i}^1 \frac{\partial w_i}{\partial a_i}(\bar{\phi}_i(s_{-i}), \tilde{s}_i, s_{-i})\frac{f_i(\tilde{s}_i)}{1 - F_i(s_i)}\mathrm{d}\tilde{s}_i, \quad \forall s_i \in [\bar{\phi}_i(s_{-i}), 1],$$

with equality at $\bar{\phi}_i(s_{-i})$.

(C3′) If $\bar{\phi}_i(s_{-i}) = 1$, $\frac{\partial w_i}{\partial a_i}(1, 1, s_{-i}) \geq 0$.

The conditions in Theorem 3 correspond to conditions c1 - c3$'$ in Amador and Bagwell (2013). In fact, our conditions are the contingent versions of theirs, as is indicated by the fact that all these conditions are indexed by s_{-i}. A key reason that we can obtain this contingent version is because in our setting, DSIC constraints for agent i can be expressed as a series of independent single-agent IC constraints, indexed by s_{-i}.

The proof follows a similar line of arguments as in Amador and Bagwell (2013). For the sake of space, we leave it to the online appendix. The main idea is to use the Lagrange method to transform the original constrained optimization problem (1) into a relaxed unconstrained problem. The major task is to show that the candidate mechanism is a solution to this relaxed problem, which in turn implies that it is also a solution to the original one. In doing so, one major step involves proving the concavity of the objective function of this relaxed problem, and this step is where the presence of the coordination payoff, i.e., u_0, which is absent in single-agent settings, causes a difficulty. To deal with this difficulty, the trick is to make condition C1 more demanding than its counterpart in Amador and Bagwell (2013). This is done through the construction of κ_i. In fact, if we defined κ_i as $\inf_{a_i,s_i\in[0,1]} -\frac{\partial^2 w_i}{\partial a_i^2}(a_i, s_i, s_{-i})$ as in Amador and Bagwell (2013), it would guarantee that the objective function is concave in *each agent's decision rule*, which in turn would imply that the interval delegations ϕ_1 and ϕ_2 are a "mutual best response." However, it is not enough to guarantee that the objective function is concave as a function of *the pair of agents' decision rules*, which, in contrast, can be guaranteed by the smaller κ_i we give in (31). Clearly, this smaller κ_i makes condition C1 more demanding since F_i is increasing.[29]

D.2 Proof of Theorem 2

Proof. For notational simplicity, we write $a_i^*(s_i, s_{-i})$, instead of $\sigma_i^{\phi_i^*}(s_i, s_{-i})$, to denote i's decision under (ϕ_1^*, ϕ_2^*). It is easy to notice that, for every s_{-i}, $a_{-i}^*(s_i, s_{-i})$ is a piecewise function in s_i: it partitions $[0, 1]$ into finitely many intervals, and over each interval it is either a constant, c_{-i}^*, or d_{-i}^*. The proof of Lemma 15 shows that both

[29] Another major step in proving that the candidate mechanism is a solution to the relaxed problem is to show that there is no profitable local deviation around the candidate mechanism. It turns out that we do not need to worry about joint local deviations. This is because the local effect can be captured by the Gateaux derivative, which can be expressed as the integration of partial derivatives with respect to a, after changing the order of the derivative and integration. When considering the partial derivative of a_i, a_{-i} is treated as given. Thus, we only need to deal with unilateral local deviation.

c^*_{-i} and d^*_{-i} are differentiable for all but at most one point, and $c^{*'}_{-i} < 1$ and $d^{*'}_{-i} < 1$. Hence, $a^*_{-i}(s_i, s_{-i})$ is differentiable with respect to s_i for all but at most finitely many points, and $\frac{\partial a^*_{-i}}{\partial s_i}(s_i, s_{-i}) < 1$. Recall that we have explained in Section 3.4 that (9) must hold when $\underline{\phi}^*_i(s_{-i}) > 0$. Using notation a^*_{-i}, we can rewrite it as

$$\int_0^{\underline{\phi}^*_i(s_{-i})} \left[\frac{\partial u_0}{\partial a_i}(\underline{\phi}^*_i(s_{-i}), a^*_{-i}(\underline{\phi}^*_i(s_{-i}), s_{-i})) + \frac{\partial u_i}{\partial a_i}(\underline{\phi}^*_i(s_{-i}), s_i)\right] \mathrm{d}F_i(s_i) = 0. \tag{32}$$

Similarly, when $\bar{\phi}^*_i(s_{-i}) < 1$, we have

$$\int_{\bar{\phi}^*_i(s_{-i})}^1 \left[\frac{\partial u_0}{\partial a_i}(\bar{\phi}^*_i(s_{-i}), a^*_{-i}(\bar{\phi}^*_i(s_{-i}), s_{-i})) + \frac{\partial u_i}{\partial a_i}(\bar{\phi}^*_i(s_{-i}), s_i)\right] \mathrm{d}F_i(s_i) = 0. \tag{33}$$

With these preparations, we are ready to verify that conditions C1 - C3′ in Theorem 3 are all satisfied under the proposed conditions. Then, by Theorem 3, we know (ϕ^*_1, ϕ^*_2) is optimal.

Step 1: Conditions C2′ and C3′ hold.

We only show condition C2′. Condition C3′ is analogous. These two conditions will be used in the verification of condition C1 below.

Fix s_{-i} such that $\underline{\phi}^*_i(s_{-i}) = 0$. Suppose, by contradiction, $\frac{\partial w_i}{\partial a_i}(0, 0, s_{-i}) > 0$. By continuity, there exists $\hat{c} \in (0,\ \bar{\phi}^*_i(s_{-i}))$ such that $\frac{\partial w_i}{\partial a_i}(c, s_i, s_{-i}) > 0$ for all $c, s_i \in [0, \hat{c}]$. Thus, for all $c \in [0, \hat{c}]$, we have

$$\int_0^c \frac{\partial w_i}{\partial a_i}(c, s_i, s_{-i}) f_i(s_i) \mathrm{d}s_i = \int_0^c \left(\frac{\partial u_i}{\partial a_i}(c, s_i) + \frac{\partial u_0}{\partial a_i}(c, a^*_{-i}(s_i, s_{-i}))\right) f_i(s_i) \mathrm{d}s_i > 0,$$

which in turn implies that, given s_{-i} and a^*_{-i}, $[\hat{c},\ \bar{\phi}^*_i(s_{-i})]$ is a better delegation interval than $[\underline{\phi}^*_i(s_{-i}),\ \bar{\phi}^*_i(s_{-i})]$ for the principal. This contradicts Theorem 1. Hence, we must have $\frac{\partial w_i}{\partial a_i}(0, 0, s_{-i}) \leq 0$.

Step 2: Condition C1 holds.

Fix s_{-i}. We want to show that

$$\kappa_i F_i(s_i) - f_i(s_i) \frac{\partial w_i}{\partial a_i}(s_i, s_i, s_{-i}) = \kappa_i F_i(s_i) - f_i(s_i) \left[\frac{\partial u_i}{\partial a_i}(s_i, s_i) + \frac{\partial u_0}{\partial a_i}(s_i, a^*_{-i}(s_i, s_{-i}))\right]$$

is increasing over $s_i \in [\underline{\phi}^*_i(s_{-i}),\ \bar{\phi}^*_i(s_{-i})]$. From condition O1, it suffices to show that $\kappa_i F_i(s_i) - f_i(s_i) \frac{\partial u_0}{\partial a_i}(s_i, a^*_{-i}(s_i, s_{-i}))$ is increasing. For every s_i at which $a^*_{-i}(s_i, s_{-i})$ is differentiable, we have

$$
\begin{aligned}
&\frac{\partial\left(\kappa_i F_i(s_i) - f_i(s_i)\frac{\partial u_0}{\partial a_i}(s_i, a^*_{-i}(s_i, s_{-i}))\right)}{\partial s_i} \\
=&\kappa_i f_i(s_i) - f'_i(s_i)\frac{\partial u_0}{\partial a_i}(s_i, a^*_{-i}(s_i, s_{-i})) \\
&- f_i(s_i)\left(\frac{\partial^2 u_0}{\partial a_i^2}(s_i, a^*_{-i}(s_i, s_{-i})) + \frac{\partial^2 u_0}{\partial a_i \partial a_{-i}}(s_i, a^*_{-i}(s_i, s_{-i}))\frac{\partial a^*_{-i}}{\partial s_i}(s_i, s_{-i})\right).
\end{aligned} \tag{34}
$$

Observe that

$$
\begin{aligned}
&\frac{\partial^2 u_0}{\partial a_i^2}(s_i, a^*_{-i}(s_i, s_{-i})) + \frac{\partial^2 u_0}{\partial a_i \partial a_{-i}}(s_i, a^*_{-i}(s_i, s_{-i}))\frac{\partial a^*_{-i}}{\partial s_i}(s_i, s_{-i}) \\
\leq&\frac{\partial^2 u_0}{\partial a_i^2}(s_i, a^*_{-i}(s_i, s_{-i})) + \frac{\partial^2 u_0}{\partial a_i \partial a_{-i}}(s_i, a^*_{-i}(s_i, s_{-i})) \leq 0,
\end{aligned} \tag{35}
$$

where the first inequality comes from $\frac{\partial^2 u_0}{\partial a_1 \partial a_2} \geq 0$ and $\frac{\partial a^*_{-i}(s_i, s_{-i})}{\partial s_i} \leq 1$. The second inequality comes from condition R2. Hence, to show that (34) is nonnegative, it suffices to show that

$$
\frac{f'_i(s_i)}{f_i(s_i)}\frac{\partial u_0}{\partial a_i}(s_i, a^*_{-i}(s_i, s_{-i})) \leq \kappa_i, \quad \forall s_i \in [\underline{\phi}^*_i(s_{-i}), \bar{\phi}^*_i(s_{-i})].
$$

If $\frac{f'_i(s_i)}{f_i(s_i)}$ and $\frac{\partial u_0}{\partial a_i}(s_i, a^*_{-i}(s_i, s_{-i}))$ have different signs, the desired inequality is obvious because $\frac{f'_i(s_i)}{f_i(s_i)}\frac{\partial u_0}{\partial a_i}(s_i, a^*_{-i}(s_i, s_{-i})) \leq 0 \leq \kappa_i$. We now consider the cases where these two terms have the same sign.

First, suppose $\frac{f'_i(s_i)}{f_i(s_i)} > 0$ and $\frac{\partial u_0}{\partial a_i}(s_i, a^*_{-i}(s_i, s_{-i})) > 0$. Because f_i is log-concave, we have $\frac{f'_i(\underline{\phi}^*_i(s_{-i}))}{f_i(\underline{\phi}^*_i(s_{-i}))} \geq \frac{f'_i(s_i)}{f_i(s_i)} > 0$. Because of (35), we know $s_i \mapsto \frac{\partial u_0}{\partial a_i}(s_i, a^*_{-i}(s_i, s_{-i}))$ is decreasing. Thus, we have $\frac{\partial u_0}{\partial a_i}(\underline{\phi}^*_i(s_{-i}), a^*_{-i}(\underline{\phi}^*_i(s_{-i}), s_{-i})) \geq \frac{\partial u_0}{\partial a_i}(s_i, a^*_{-i}(s_i, s_{-i})) > 0$. These inequalities have two implications. First, we have

$$
\frac{f'_i(s_i)}{f_i(s_i)}\frac{\partial u_0}{\partial a_i}(s_i, a^*_{-i}(s_i, s_{-i})) \leq \frac{f'_i(\underline{\phi}^*_i(s_{-i}))}{f_i(\underline{\phi}^*_i(s_{-i}))}\frac{\partial u_0}{\partial a_i}(\underline{\phi}^*_i(s_{-i}), a^*_{-i}(\underline{\phi}^*_i(s_{-i}), s_{-i})). \tag{36}
$$

Second, we have $\underline{\phi}^*_i(s_{-i}) > 0$. To see this, suppose by contradiction, that $\underline{\phi}^*_i(s_{-i}) = 0$. Then $\frac{f'_i(\underline{\phi}^*_i(s_{-i}))}{f_i(\underline{\phi}^*_i(s_{-i}))} > 0$ implies $\frac{\partial u_i}{\partial a_i}(0,0) \geq 0$ by condition O2. But then $\frac{\partial u_0}{\partial a_i}(0, a^*_{-i}(0, s_{-i})) + \frac{\partial u_i}{\partial a_i}(0,0) > 0$, contradicting condition C2′. Thus, we can only have $\underline{\phi}^*_i(s_{-i}) > 0$. Then, from (32), we have

$$\begin{aligned}
&\frac{\partial u_0}{\partial a_i}(\underline{\phi}_i^*(s_{-i}), a_{-i}^*(\underline{\phi}_i^*(s_{-i}), s_{-i}))\\
=&\frac{1}{F_i(\underline{\phi}_i^*(s_{-i}))}\int_0^{\underline{\phi}_i^*(s_{-i})}\Big(-\frac{\partial u_i}{\partial a_i}(\underline{\phi}_i^*(s_{-i}), s_i)\Big)f_i(s_i)\mathrm{d}s_i\\
\leq&\frac{1}{F_i(\underline{\phi}_i^*(s_{-i}))}\int_0^{\underline{\phi}_i^*(s_{-i})}\Big(\frac{\partial u_i}{\partial a_i}(\underline{\phi}_i^*(s_{-i}), \underline{\phi}_i^*(s_{-i}))-\frac{\partial u_i}{\partial a_i}(\underline{\phi}_i^*(s_{-i}), s_i)\Big)f_i(s_i)\mathrm{d}s_i\\
=&\frac{1}{F_i(\underline{\phi}_i^*(s_{-i}))}\int_0^{\underline{\phi}_i^*(s_{-i})}\Big(\int_{s_i}^{\underline{\phi}_i^*(s_{-i})}\frac{\partial^2 u_i}{\partial a_i\partial s_i}(\underline{\phi}_i^*(s_{-i}), x)\mathrm{d}x\Big)f_i(s_i)\mathrm{d}s_i\\
=&\frac{1}{F_i(\underline{\phi}_i^*(s_{-i}))}\int_0^{\underline{\phi}_i^*(s_{-i})}\frac{\partial^2 u_i}{\partial a_i\partial s_i}(\underline{\phi}_i^*(s_{-i}), x)F_i(x)\mathrm{d}x\\
\leq&\frac{\kappa_i}{F_i(\underline{\phi}_i^*(s_{-i}))}\int_0^{\underline{\phi}_i^*(s_{-i})}F_i(x)\mathrm{d}x, \qquad (37)
\end{aligned}$$

where the first inequality comes from $\frac{\partial u_0}{\partial a_i}(\underline{\phi}_i^*(s_{-i}), \underline{\phi}_i^*(s_{-i})) \geq 0$ by condition O2. The second inequality comes from condition O3. Combining (36) and (37) yields

$$\frac{f_i'(s_i)}{f_i(s_i)}\frac{\partial u_0}{\partial a_i}(s_i, a_{-i}^*(s_i, s_{-i})) \leq \kappa_i\frac{f_i'(\underline{\phi}_i^*(s_{-i}))}{f_i(\underline{\phi}_i^*(s_{-i}))}\frac{\int_0^{\underline{\phi}_i^*(s_{-i})}F_i(x)\mathrm{d}x}{F_i(\underline{\phi}_i^*(s_{-i}))} \leq \kappa_i,$$

where the last inequality comes from part (ii) of Lemma 16.

Next, suppose $\frac{f_i'(s_i)}{f_i(s_i)} < 0$ and $\frac{\partial u_0}{\partial a_i}(s_i, a_{-i}^*(s_i, s_{-i})) < 0$. Similarly as above, we have $\frac{f_i'(\bar{\phi}_i^*(s_{-i}))}{f_i(\bar{\phi}_i^*(s_{-i}))} \leq \frac{f_i'(s_i)}{f_i(s_i)} < 0$ and $\frac{\partial u_0}{\partial a_i}(\bar{\phi}_i^*(s_{-i}), a_{-i}^*(\bar{\phi}_i^*(s_{-i}), s_{-i})) \leq \frac{\partial u_0}{\partial a_i}(s_i, a_{-i}^*(s_i, s_{-i})) < 0$. Thus, we have

$$\frac{f_i'(s_i)}{f_i(s_i)}\frac{\partial u_0}{\partial a_i}(s_i, a_{-i}^*(s_i, s_{-i})) \leq \frac{f_i'(\bar{\phi}_i^*(s_{-i}))}{f_i(\bar{\phi}_i^*(s_{-i}))}\frac{\partial u_0}{\partial a_i}(\bar{\phi}_i^*(s_{-i}), a_{-i}^*(\bar{\phi}_i^*(s_{-i}), s_{-i})). \qquad (38)$$

Moreover, we also have $\bar{\phi}_i^*(s_{-i}) < 1$. Thus, using the first order condition (33) and applying conditions O2 and O3 as above, we can similarly show that

$$\frac{\partial u_0}{\partial a_i}(\bar{\phi}_i^*(s_{-i}), a_{-i}^*(\bar{\phi}_i^*(s_{-i}), s_{-i})) \geq \frac{-\kappa_i}{1-F_i(\bar{\phi}_i^*(s_{-i}))}\int_{\bar{\phi}_i^*(s_{-i})}^1(1-F_i(x))\mathrm{d}x. \qquad (39)$$

Combining (38) and (39) yields

$$\frac{f_i'(s_i)}{f_i(s_i)}\frac{\partial u_0}{\partial a_i}(s_i, a_{-i}^*(s_i, s_{-i})) \leq \kappa_i\frac{-f_i'(\bar{\phi}_i^*(s_{-i}))}{f_i(\bar{\phi}_i^*(s_{-i}))}\frac{\int_{\bar{\phi}_i^*(s_{-i})}^1(1-F_i(x))\mathrm{d}x}{1-F_i(\bar{\phi}_i^*(s_{-i}))} \leq \kappa_i,$$

where the last inequality comes again from part (ii) of Lemma 16.

Step 3: Conditions C2 and C3 hold.

We only show condition C2. Condition C3 is similar. Fix s_{-i} such that $\underline{\phi}_i^*(s_{-i}) > 0$. Let

$$g(s_i) \equiv (s_i - \underline{\phi}_i^*(s_{-i}))\kappa_i - \int_0^{s_i} \frac{\partial w_i}{\partial a_i}(\underline{\phi}_i^*(s_{-i}), \tilde{s}_i, s_{-i}) \frac{f_i(\tilde{s}_i)}{F_i(s_i)} \mathrm{d}\tilde{s}_i, \ \forall s_i \in [0,\ \underline{\phi}_i^*(s_{-i})].$$

It is straightforward to see that the first order condition (9) directly implies $g(\underline{\phi}_i^*(s_{-i})) = 0$. Hence, to show C2, it suffices to show that $g'(s_i) \geq 0$ for $s_i \in [0,\ \underline{\phi}_i^*(s_{-i})]$. We can calculate

$$g'(s_i) = \kappa_i - \frac{f_i(s_i)}{F_i^2(s_i)} \int_0^{s_i} \left[\frac{\partial w_i}{\partial a_i}(\underline{\phi}_i^*(s_{-i}), s_i, s_{-i}) - \frac{\partial w_i}{\partial a_i}(\underline{\phi}_i^*(s_{-i}), \tilde{s}_i, s_{-i})\right] f_i(\tilde{s}_i)\mathrm{d}\tilde{s}_i.$$

Recall that

$$\frac{\partial w_i}{\partial a_i}(a_i, s_i, s_{-i}) = \frac{\partial u_i}{\partial a_i}(a_i, s_i) + \frac{\partial u_0}{\partial a_i}(a_i, a_{-i}^*(s_i, s_{-i})).$$

Because $a_{-i}^*(s_i, s_{-i}) = a_{-i}^*(\underline{\phi}_i^*(s_{-i}), s_{-i})$ for all $s_i \leq \underline{\phi}_i^*(s_{-i})$ as explained previously, we know

$$\frac{\partial w_i}{\partial a_i}(\underline{\phi}_i^*(s_{-i}), s_i, s_{-i}) - \frac{\partial w_i}{\partial a_i}(\underline{\phi}_i^*(s_{-i}), \tilde{s}_i, s_{-i}) = \frac{\partial u_i}{\partial a_i}(\underline{\phi}_i^*(s_{-i}), s_i) - \frac{\partial u_i}{\partial a_i}(\underline{\phi}_i^*(s_{-i}), \tilde{s}_i),$$

implying

$$\begin{aligned} g'(s_i) &= \kappa_i - \frac{f_i(s_i)}{F_i^2(s_i)} \int_0^{s_i} \int_{\tilde{s}_i}^{s_i} \frac{\partial^2 u_i}{\partial a_i \partial s_i}(\underline{\phi}_i(s_{-i}), x, s_{-i}) f_i(\tilde{s}_i)\mathrm{d}x\mathrm{d}\tilde{s}_i \\ &\geq \kappa_i \left(1 - \frac{f_i(s_i)}{F_i^2(s_i)} \int_0^{s_i} \int_{\tilde{s}_i}^{s_i} f_i(\tilde{s}_i)\mathrm{d}x\mathrm{d}\tilde{s}_i\right) = \kappa_i \left(1 - \frac{f_i(s_i)}{F_i^2(s_i)} \int_0^{s_i} F_i(x)\mathrm{d}x\right) \geq 0, \end{aligned}$$

where the first inequality comes from condition O3. The last inequality comes again from part (ii) of Lemma 16. □

References

Alonso, R., I. Brocas, and J. D. Carrillo (2014): "Resource Allocation in the Brain," *Review of Economic Studies*, 81, 501–534.

Alonso, R., W. Dessein, and N. Matouschek (2008): "When Does Coordination Require Centralization?" *American Economic Review*, 98, 145–179.

Alonso, R. and N. Matouschek (2008): "Optimal Delegation," *Review of Economic Studies*, 75, 259–293.

Amador, M. and K. Bagwell (2013): "The Theory of Optimal Delegation with an Application to Tariff Caps," *Econometrica*, 81, 1541–1599.

Amador, M., K. Bagwell, and A. Frankel (2018): "A Note on Interval Delegation," *Economic Theory Bulletin*, 239–249.

An, M. Y. (1998): "Logconcavity versus Logconvexity: A Complete Characterization," *Journal of Economic Theory*, 80, 350–369.

Aoki, M. (1986): "Horizontal vs. Vertical Information Structure of the Firm," *American Economic Review*, 971–983.

Armstrong, M. (1995): "Delegation and Discretion," Mimeo, University College London.

Bagnoli, M. and T. Bergstrom (2005): "Log-concave Probability and its Applications," *Economic Theory*, 26, 445–469.

Cremer, J., L. Garicano, and A. Prat (2007): "Language and the Theory of the Firm," *Quarterly Journal of Economics*, 122, 373–407.

Dessein, W., L. Garicano, and R. Gertner (2010): "Organizing for Synergies," *American Economic Journal: Microeconomics*, 2, 77–114.

Dessein, W. and T. Santos (2006): "Adaptive Organizations," *Journal of Political Economy*, 114, 956–995.

Friebel, G. and M. Raith (2010): "Resource Allocation and Organizational Form," *American Economic Journal: Microeconomics*, 2, 1–33.

Fuchs, W., S. Fukuda, and M. Sefidgaran (2022): "Who to Listen to?: A Model of Endogenous Delegation," CEPR discussion paper no. dp17319.

Gershkov, A., J. K. Goeree, A. Kushnir, B. Moldovanu, and X. Shi (2013): "On the Equivalence of Bayesian and Dominant Strategy Implementation," *Econometrica*, 81, 197–220.

Hart, O. and J. Moore (2005): "On the Design of Hierarchies: Coordination versus Specialization," *Journal of Political Economy*, 113, 675–702.

Holmström, B. (1977): "On Incentives and Control in Organizations," Ph.D. thesis, Stanford University.

——— (1984): "On The Theory of Delegation," in *Bayesian Models in Economic Theory*, ed. by M. Boyer and R. Kihlstrom, North-Holland.

Jarman, F. and V. Meisner (2017): "Deterministic Mechanisms, the Revelation Principle, and ex-post Constraints," *Economics Letters*, 161, 96–98.

Kovac, E. and T. Mylovanov (2009): "Stochastic Mechanisms in Settings without Monetary Transfers: The Regular Case," *Journal of Economic Theory*, 144, 1373–1395.

Li, S. and X. Weng (2017): "Random Authority," *International Economic Review*, 58, 211–235.

Martimort, D. and A. Semenov (2006): "Continuity in Mechanism Design without Transfers," *Economics Letters*, 93, 182–189.

——— (2008): "The Informational Effects of Competition and Collusion in Legislative Politics," *Journal of Public Economics*, 92, 1541–1563.

Melumad, N. D. and T. Shibano (1991): "Communication in Settings with No Transfers," *RAND Journal of Economics*, 22, 173–198.

Rantakari, P. (2008): "Governing Adaptation," *Review of Economic Studies*, 75, 1257–1285.

Roberts, J. (2004): *The Modern Firm*, Oxford University Press.

Rudin, W. (1976): *Principles of Mathematical Analysis*, McGraw-Hill New York.

Shaked, M. and J. G. Shanthikumar (2007): *Stochastic Orders*, Springer Science & Business Media.

Topkis, D. M. (1998): *Supermodularity and Complementarity*, Princeton University Press.

Online Appendix for "Optimal Contingent Delegation"

Tan Gan, Ju Hu and Xi Weng

October 5, 2022

This online appendix contains missing proofs. Section A provides the missing proof of Lemma 12. Section B provides the proof of Theorem 3 in Appendix D.1. Section C contains the proofs for Section 4.

Online Appendix A Missing Proof of Lemma 12

In Appendix B.3, we have proved Lemma 12 assuming that there exist desired h_1 and h_2 that satisfy parts (i) and (ii) of Lemma 12. The next lemma confirms the existence of such h_1 and h_2.

Lemma A.1. *For every $s_1 \in [\underline{L}_1, \bar{H}_1]$, there exists a unique $h_2(s_1) \in [c_2^*(s_1),\, d_2^*(s_1)]$ such that the following equation holds*

$$s_1 = \frac{h_2(s_1) - c_2^*(s_1)}{d_2^*(s_1) - c_2^*(s_1)} d_1^*(h_2(s_1)) + \frac{d_2^*(s_1) - h_2(s_1)}{d_2^*(s_1) - c_2^*(s_1)} c_1^*(h_2(s_1)). \tag{A.1}$$

Then, $h_1 \equiv h_2^{-1}$ and h_2 satisfy parts (i) and (ii) of Lemma 12.

Proof. For every $s_1 \in [\underline{L}_1, \bar{H}_1]$ and $s_2 \in [c_2^*(s_1),\, d_2^*(s_1)]$, define

$$g(s_1, s_2) \equiv \frac{s_2 - c_2^*(s_1)}{d_2^*(s_1) - c_2^*(s_1)} d_1^*(s_2) + \frac{d_2^*(s_1) - s_2}{d_2^*(s_1) - c_2^*(s_1)} c_1^*(s_2). \tag{A.2}$$

It is well defined by condition U and continuous by Lemma 2. We divide the remaining proof into several small steps.

Step 1: For every s_1, $g(s_1, \cdot\,)$ is strictly increasing.

Consider $c_2^*(s_1) \le s_2 < s_2' \le d_2^*(s_1)$. We have

$$\begin{aligned} g(s_1, s_2) &\le \frac{s_2 - c_2^*(s_1)}{d_2^*(s_1) - c_2^*(s_1)} d_1^*(s_2') + \frac{d_2^*(s_1) - s_2}{d_2^*(s_1) - c_2^*(s_1)} c_1^*(s_2') \\ &= \frac{s_2 - c_2^*(s_1)}{d_2^*(s_1) - c_2^*(s_1)} (d_1^*(s_2') - c_1^*(s_2')) + c_1^*(s_2') \\ &< \frac{s_2' - c_2^*(s_1)}{d_2^*(s_1) - c_2^*(s_1)} (d_1^*(s_2') - c_1^*(s_2')) + c_1^*(s_2') \\ &= g(s_1, s_2'), \end{aligned}$$

where the first inequality comes from monotonicity of c_1^* and d_1^* by Lemma 2. The second inequality comes from $d_1^*(s_2') > c_1^*(s_2')$ by condition U.

Step 2: If $s_1 = \underline{L}_1$, the unique $h_2(s_1) \in [c_2^*(\underline{L}_1),\ d_2^*(\underline{L}_1)]$ that satisfies $g(s_1, h_2(s_1)) = s_1$ is $h_2(s_1) = \underline{L}_2$.

Because $c_2^*(\underline{L}_1) = \underline{L}_2$ and $c_1^*(\underline{L}_2) = \underline{L}_1$, it is straightforward to see $g(\underline{L}_1, \underline{L}_2) = \underline{L}_1$. Uniqueness comes from the previous step.

Step 3: If $s_1 = \bar{H}_1$, the unique $h_2(s_1) \in [c_2^*(\bar{H}_1),\ d_2^*(\bar{H}_1)]$ that satisfies $g(s_1, h_2(s_1)) = s_1$ is $h_2(s_1) = \bar{H}_2$.

The proof is similar to the previous one.

Step 4: If $s_1 \in (\underline{L}_1,\ \bar{H}_1)$, then there exists a unique $h_2(s_1) \in (c_2^*(s_1),\ d_2^*(s_1))$ such that $g(s_1, h_2(s_1)) = s_1$.

It is easy to see $g(s_1, c_2^*(s_1)) = c_1^*(c_2^*(s_1))$. Because $s_1 > \underline{L}_1$, we then know $g(s_1, c_2^*(s_1)) < s_1$ by Lemma 9. Similarly, because $g(s_1, d_2^*(s_1)) = d_1^*(d_2^*(s_1))$ and $s_1 < \bar{H}_1$, we know $g(s_1, d_2^*(s_1)) > s_1$ by Lemma 9 again. Thus, by Step 1, we know there exists a unique $h_2(s_1) \in (c_2^*(s_1),\ d_2^*(s_1))$ such that $g(s_1, h_2(s_1)) = s_1$.

Step 5: $h_2 : [\underline{L}_1,\ \bar{H}_1] \to [\underline{L}_2,\ \bar{H}_2]$ is continuous and surjective.

Let $\{s_1^n\}_{n\geq 1} \subset [\underline{L}_1, \bar{H}_1]$ be a sequence converging to $s_1 \in [\underline{L}_1, \bar{H}_1]$. Because $\{h_2(s_1^n)\}_{n\geq 1} \subset [\underline{L}_2, \bar{H}_2]$, it has a convergent subsequence $\{h_2(s_1^{n_k})\}_{k\geq 1}$. Let $s_2 \equiv \lim_{k\to\infty} h_2(s_1^{n_k}) \in [c_2^*(s_1),\ d_2^*(s_1)]$. Because $g(s_1^{n_k}, h_2(s_1^{n_k})) = s_1^{n_k}$ for all $k \geq 1$ and g is continuous, we know $g(s_1, s_2) = s_1$. By Steps 2 - 4, we know $s_2 = h_2(s_1)$. This proves the continuity of h_2. Because $h_2(\underline{L}_1) = \underline{L}_2$ and $h_2(\bar{H}_1) = \bar{H}_2$ by Steps 2 and 3, we know h_2 is surjective since it is continuous.

Step 6: $h_2(\underline{L}_1) < h_2(s_1) < h_2(\bar{H}_1)$ for all $s_1 \in (\underline{L}_1, \bar{H}_1)$.

For all $s_1 \in (\underline{L}_1, \bar{H}_1)$, we have

$$h_2(\underline{L}_1) = \underline{L}_2 = c_2^*(\underline{L}_1) \leq c_2^*(s_1) < h_2(s_1) < d_2^*(s_1) \leq d_2^*(\bar{H}_1) = \bar{H}_2 = h_2(\bar{H}_1),$$

where the first and last equalities come from Steps 2 and 3. The two weak inequalities come from monotonicity of c_2^* and d_2^*. The two strict inequalities come from Step 4.

Step 7: $h_2 : [\underline{L}_1, \bar{H}_1] \to [\underline{L}_2, \bar{H}_2]$ is strictly increasing.

We first argue that h_2 is injective. Consider $\underline{L}_1 \leq s_1 < s_1' \leq \bar{H}_1$. Suppose, by contradiction, $h_2(s_1) = h_2(s_1') \equiv s_2$. By Step 6, we know $\underline{L}_1 < s_1 < s_1' < \bar{H}_1$. Thus, $c_2^*(s_1) < s_2 < d_2^*(s_1)$ and $c_2^*(s_1') < s_2 < d_2^*(s_1')$ by Step 4.

Because $g(s_1, s_2) = s_1 < s_1' = g(s_1', s_2)$ and $d_1^*(s_2) > c_1^*(s_2)$, we can directly see from (A.2) that

$$\frac{s_2 - c_2^*(s_1)}{d_2^*(s_1) - c_2^*(s_1)} < \frac{s_2 - c_2^*(s_1')}{d_2^*(s_1') - c_2^*(s_1')},$$

which implies

$$\frac{d_2^*(s_1) - s_2}{s_2 - c_2^*(s_1)} > \frac{d_2^*(s_1') - s_2}{s_2 - c_2^*(s_1')}.$$

But this is impossible, since $0 < s_2 - c_2^*(s_1') \leq s_2 - c_2^*(s_1')$ and $0 < d_2^*(s_1) - s_2 \leq d_2^*(s_1') - s_2$. Therefore, h_2 is injective.

Because h_2 is continuous by Step 5, we now know h_2 is strictly monotone. Because $h_2(\underline{L}_1) < h_2(\bar{H}_1)$, we know h_2 is strictly increasing.

The above Steps 2 - 4 and 7 together guarantee that h_2 satisfies parts (i) and (ii) in Lemma 12. These steps, together with Step 5, guarantee that $h_1 \equiv h_2^{-1} : [\underline{L}_2, \bar{H}_2] \to [\underline{L}_1, \bar{H}_1]$ is well defined and satisfies part (i).

Step 8: For all $s_2 \in (\underline{L}_2, \bar{H}_2)$, $h_1(s_1) \in (c_1^*(s_2), d_1^*(s_2))$. That is, h_1 satisfies part (ii).

Let $s_1 \equiv h_1(s_2) \in (\underline{L}_1, \bar{H}_1)$. Then, (A.1) can be written as

$$h_1(s_2) = \frac{h_2(s_1) - c_2^*(s_1)}{d_2^*(s_1) - c_2^*(s_1)} d_1^*(s_2) + \frac{d_2^*(s_1) - h_2(s_1)}{d_2^*(s_1) - c_2^*(s_1)} c_1^*(s_2).$$

Because $\frac{h_2(s_1) - c_2^*(s_1)}{d_2^*(s_1) - c_2^*(s_1)} \in (0, 1)$ by Step 4, we immediately know $h_1(s_2) \in (c_1^*(s_2), d_1^*(s_2))$. This completes the proof. $\square$

Online Appendix B Proof of Theorem 3

Proof of Theorem 3. For notational simplicity, we write $a_i^*(s_i, s_{-i})$ for $\sigma_i^\phi(s_i, s_{-i})$. The goal is to show that $a^* \equiv (a_1^*, a_2^*)$ solves the following problem, which is equivalent to (1) by the standard envelope theorem argument:

$$\max_{(a_1,a_2)} \iint \Big(u_0(a_1(s_1, s_2), a_2(s_1, s_2)) + \sum_i u_i(a_i(s_i, s_{-i}), s_i)\Big) f_1(s_1) f_2(s_2)\, \mathrm{d}s_1 \mathrm{d}s_2, \tag{B.1}$$

subject to:

$$s_i a_i(s_i, s_{-i}) - \frac{a_i(s_i, s_{-i})^2}{2} = \int_0^{s_i} a_i(\tilde{s}_i, s_{-i}) \mathrm{d}\tilde{s}_i - \frac{a_i(0, s_{-i})^2}{2}, \ \forall i, s_i, s_{-i},$$

$a_i(s_i, s_{-i})$ is increasing in s_i, $\forall i, s_{-i}$.

Define the following (cumulative) Lagrange multiplier:

$$\Lambda_i(s_i, s_{-i}) = \begin{cases} f_{-i}(s_{-i})(1 - \kappa_i F_i(s_i)), & s_i \in [0,\, \underline{\phi}_i(s_{-i})], \\ f_{-i}(s_{-i})(1 - \frac{\partial w_i}{\partial a_i}(s_i, s_i, s_{-i}) f_i(s_i)), & s_i \in (\underline{\phi}_i(s_{-i}),\, \bar{\phi}_i(s_{-i})), \\ f_{-i}(s_{-i})(1 + \kappa_i(1 - F_i(s_i))), & s_i \in [\bar{\phi}_i(s_{-i}),\, 1]. \end{cases}$$

We argue that, for every s_{-i}, the following function is increasing in s_i:

$$\begin{aligned} &\Lambda_i(s_i, s_{-i}) + \kappa_i f_{-i}(s_{-i}) F_i(s_i) \\ =& \begin{cases} f_{-i}(s_{-i}), & s_i \in [0,\, \underline{\phi}_i(s_{-i})], \\ f_{-i}(s_{-i})(1 + \kappa_i F_i(s_i) - \frac{\partial w_i}{\partial a_i}(s_i, s_i, s_{-i}) f_i(s_i)), & s_i \in (\underline{\phi}_i(s_{-i}),\, \bar{\phi}_i(s_{-i})), \\ f_{-i}(s_{-i})(1 + \kappa_i), & s_i \in [\bar{\phi}_i(s_{-i}),\, 1], \end{cases} \end{aligned}$$

Clearly, it is increasing over $[0,\, \underline{\phi}_i(s_{-i})]$ and $[\bar{\phi}_i(s_{-i}),\, 1]$. By condition C1, it is also increasing over $[\underline{\phi}_i(s_{-i}),\, \bar{\phi}_i(s_{-i})]$. Hence, to show that it is increasing over $[0, 1]$, it suffices to verify the following two inequalities:

$$\kappa_i F_i(\underline{\phi}_i(s_{-i})) \geq \frac{\partial w_i}{\partial a_i}(\underline{\phi}_i(s_{-i}), \underline{\phi}_i(s_{-i}), s_{-i}) f_i(\underline{\phi}_i(s_{-i})), \tag{B.2}$$

$$\kappa_i(1 - F_i(\bar{\phi}_i(s_{-i}))) \geq -\frac{\partial w_i}{\partial a_i}(\bar{\phi}_i(s_{-i}), \bar{\phi}_i(s_{-i}), s_{-i}) f_i(\bar{\phi}_i(s_{-i})). \tag{B.3}$$

If $\underline{\phi}_i(s_{-i}) = 0$, (B.2) is directly implied by condition C2′. If $\underline{\phi}_i(s_{-i}) > 0$, we know from condition C2 that

$$g(s_i) = (s_i - \underline{\phi}_i(s_{-i}))\kappa_i F_i(s_i) - \int_0^{s_i} \frac{\partial w_i}{\partial a_i}(\underline{\phi}_i(s_{-i}), \tilde{s}_i, s_{-i}) f_i(\tilde{s}_i) \mathrm{d}\tilde{s}_i \leq 0, \ \forall s_i \in [0, \underline{\phi}_i(s_{-i})],$$

with equality at $\underline{\phi}_i(s_{-i})$. This implies that $g'(\underline{\phi}_i(s_{-i})) \geq 0$. Equivalently, (B.2) holds. Using conditions C3 and C3′, we can similarly verify that (B.3) also holds.

For every s_{-i}, being the difference of two increasing functions, $\Lambda_i(s_i, s_{-i})$ as a function of s_i has bounded variation. As a result, it induces a well-defined (signed) measure $\Lambda_i(\mathrm{d}s_i, s_{-i})$ over $[0, 1]$. Let

$$\Phi \equiv \{\text{direct mechanism } (a_1, a_2) \,|\, a_i(s_i, s_{-i}) \text{ is increasing in } s_i\}.$$

Define the Lagrangian function $\mathcal{L} : \Phi \to \mathbb{R}$ as, for every $a \in \Phi$,

$$\begin{aligned} \mathcal{L}(a) \equiv& \iint \Big(u_0(a_1(s_1, s_2), a_2(s_1, s_2)) + \sum_i u_i(a_i(s_i, s_{-i}), s_i)\Big) f_1(s_1) f_2(s_2)\, \mathrm{d}s_1 \mathrm{d}s_2 \\ &- \sum_i \iint \Big(\int_0^{s_i} a_i(\tilde{s}_i, s_{-i}) \mathrm{d}\tilde{s}_i - \frac{a_i(0, s_{-i})^2}{2} - s_i a_i(s_i, s_{-i}) + \frac{a_i(s_i, s_{-i})^2}{2}\Big) \Lambda_i(\mathrm{d}s_i, s_{-i}) \mathrm{d}s_{-i} \end{aligned}$$

In what follows, we proceed to show that a^* solves

$$\max_{a\in\Phi} \mathcal{L}(a), \tag{B.4}$$

which is sufficient for a^* to be a solution to (B.1).

Step 1: $\mathcal{L}$ is concave.

Note that for all s_{-i},

$$\int_0^1 \Big(\int_0^{s_i} a_i(\tilde{s}_i, s_{-i})\mathrm{d}\tilde{s}_i\Big)\Lambda_i(\mathrm{d}s_i, s_{-i}) = \int_0^1 a_i(s_i, s_{-i})\big(\Lambda_i(1, s_{-i}) - \Lambda_i(s_i, s_{-i})\big)\mathrm{d}s_i,$$
$$\int_0^1 -\frac{a_i(0, s_{-i})^2}{2}\Lambda_i(\mathrm{d}s_i, s_{-i}) = -\frac{a_i(0, s_{-i})^2}{2}(\Lambda_i(1, s_{-i}) - \Lambda_i(0, s_{-i})) = 0,$$

where the last equality comes from the construction of Λ_i. Hence, $\mathcal{L}(a)$ can be rewritten as

$$\begin{aligned}
\mathcal{L}(a) = &\iint \Big(u_0(a(s))f_1(s_1)f_2(s_2) - \sum_i a_i(s)(\Lambda_i(1, s_{-i}) - \Lambda_i(s_i, s_{-i}))\Big)\mathrm{d}s_1\mathrm{d}s_2 \\
&+ \sum_i \int_0^1\int_0^1 u_i(a_i(s), s_i)f_1(s_1)f_2(s_2)\mathrm{d}s_1\mathrm{d}s_2 \\
&+ \sum_i \int_0^1\int_0^1 \Big(s_i a_i(s) - \frac{a_i(s)^2}{2}\Big)\Lambda_i(\mathrm{d}s_i, s_{-i})\mathrm{d}s_{-i} \\
= &\iint \underbrace{\Big(u_0(a(s))f_1(s_1)f_2(s_2) - \sum_i a_i(s)(\Lambda_i(1, s_{-i}) - \Lambda_i(s_i, s_{-i}))\Big)}_{A(a,s)}\mathrm{d}s_1\mathrm{d}s_2 && \text{(B.5)} \\
&+ \sum_i \int_0^1\int_0^1 \underbrace{\Big(u_i(a_i(s), s_i) - \kappa_i s_i a_i(s) + \kappa_i\frac{a_i(s)^2}{2}\Big)}_{B_i(a,s)} f_1(s_1)f_2(s_2)\mathrm{d}s_1\mathrm{d}s_2 && \text{(B.6)} \\
&+ \sum_i \int_0^1\int_0^1 \underbrace{\Big(s_i a_i(s) - \frac{a_i(s)^2}{2}\Big)}_{C_i(a,s)}(\Lambda_i(\mathrm{d}s_i, s_{-i}) + \kappa_i f_{-i}(s_{-i})F_i(\mathrm{d}s_i))\mathrm{d}s_{-i}, && \text{(B.7)}
\end{aligned}$$

where the second equality is obtained by simultaneously adding and subtracting the term $\sum_i \int_0^1\int_0^1 \big(\kappa_i s_i a_i(s_i, s_{-i}) - \kappa_i\frac{a_i(s_i,s_{-i})^2}{2}\big)f_1(s_1)f_2(s_2)\mathrm{d}s_1\mathrm{d}s_2$. For any s, $A(a, s)$ is concave in a because u_0 is concave. Hence, the integral in (B.5) is concave in a. For each i and s, $B_i(a, s)$ is also concave in a by the definition of κ_i. Hence, the term in (B.6) is concave in a. For any i and s, $C_i(a, s)$ is concave in a. Because we have already shown that $\Lambda_i(s_i, s_{-i}) + \kappa_i f_{-i}(s_{-i}F_i(s_i))$ is increasing in s_i, $\Lambda_i(\mathrm{d}s_i, s_{-i}) + \kappa_i f_{-i}(s_{-i}F_i(\mathrm{d}s_i))$ is in fact a positive measure. Hence, the term in (B.7) is also concave in a. Being the sum of functionals that are concave in a, $\mathcal{L}$ is also concave in a.

Step 2: For every $a \in \Phi$, $\lim_{\alpha\to 0} \frac{\mathcal{L}(\alpha a+(1-\alpha)a^*)-\mathcal{L}(a^*)}{\alpha} \leq 0$.

For each $a \in \Phi$, using the expression of $\mathcal{L}(a)$ in the previous step, we can directly calculate the Gateaux derivative[30]

$$
\begin{aligned}
\partial\mathcal{L}(a) \equiv& \lim_{\alpha\to 0} \frac{\mathcal{L}(a^* + \alpha a) - \mathcal{L}(a^*)}{\alpha}\\
=& \sum_i \iint \Big(\frac{\partial w_i}{\partial a_i}(a_i^*(s), s) f_1(s_1) f_2(s_2) - (\Lambda_i(1, s_{-i}) - \Lambda_i(s))\Big) a_i(s) \mathrm{d}s_1 \mathrm{d}s_2\\
&+ \sum_i \iint \Big(s_i - a_i^*(s)\Big) a_i(s) \Lambda_i(\mathrm{d}s_i, s_{-i}) \mathrm{d}s_{-i}
\end{aligned}
$$

Recall that

$$
\Lambda_i(1, s_{-i}) - \Lambda_i(s_i, s_{-i}) = \begin{cases} \kappa_i F_i(s_i) f_{-i}(s_{-i}), & \text{if } s_i \in [0, \underline{\phi}_i(s_{-i})],\\ \frac{\partial w_i}{\partial a_i}(s_i, s_i, s_{-i}) f_i(s_i) f_{-i}(s_{-i}), & \text{if } s_i \in (\underline{\phi}_i(s_{-i}), \bar{\phi}_i(s_{-i})),\\ -\kappa_i(1 - F_i(s_i)) f_{-i}(s_{-i}), & \text{if } s_i \in [\bar{\phi}_i(s_{-i}), 1], \end{cases}
$$

and

$$
a_i^*(s) = \begin{cases} \underline{\phi}_i(s_{-i}), & \text{if } s_i \in [0,\, \underline{\phi}_i(s_{-i})],\\ s_i, & \text{if } s_i \in (\underline{\phi}_i(s_{-i}),\, \bar{\phi}_i(s_{-i})),\\ \bar{\phi}_i(s_{-i}), & \text{if } s_i \in [\bar{\phi}_i(s_{-i}),\, 1]. \end{cases}
$$

Hence, we can simplify the expression of $\partial\mathcal{L}(a)$ to

$$
\begin{aligned}
&\partial\mathcal{L}(a)\\
&= \sum_i \int_0^1 \Big[\underbrace{\int_0^{\underline{\phi}_i(s_{-i})} \Big(\frac{\partial w_i}{\partial a_i}(\underline{\phi}_i(s_{-i}), s) f_i(s_i) - \kappa_i F_i(s_i) - \kappa_i(s_i - \underline{\phi}_i(s_{-i})) f_i(s_i)\Big) a_i(s) \mathrm{d}s_i}_{\ell_i(a, s_{-i})}\Big] \mathrm{d}F_{-i}\\
&+ \sum_i \int_0^1 \Big[\underbrace{\int_{\bar{\phi}_i(s_{-i})}^1 \Big(\frac{\partial w_i}{\partial a_i}(\bar{\phi}_i(s_{-i}), s) f_i(s_i) + \kappa_i(1 - F_i(s_i)) - \kappa_i(s_i - \bar{\phi}_i(s_{-i})) f_i(s_i)\Big) a_i(s) \mathrm{d}s_i}_{h_i(a, s_{-i})}\Big] \mathrm{d}F_{-i}.
\end{aligned}
$$

[30] Let $f : [0,1]^2 \to \mathbb{R}$ be a continuously differentiable function, and μ be a finite measure over $[0,1]^2$. Then,

$$
\begin{aligned}
&\lim_{\alpha\to 0} \frac{\int_{[0,1]^2} f(a^*(s) + \alpha a(s)) \mu(\mathrm{d}s) - \int_{[0,1]^2} f(a^*(s)) \mu(\mathrm{d}s)}{\alpha}\\
=& \int_{[0,1]^2} \lim_{\alpha\to 0} \frac{f(a^*(s) + \alpha a(s)) - f(a^*(s))}{\alpha} \mu(\mathrm{d}s)\\
=& \int_{[0,1]^2} \Big(\sum_i \frac{\partial f}{\partial a_i}(a^*(s)) a_i(s)\Big) \mu(\mathrm{d}s),
\end{aligned}
$$

where the first equality comes from interchanging the order of limit and integration. This is guaranteed by the bounded convergence theorem.

Consider $\ell_i(a, s_{-i})$ first. Using the fact that $a_i(s)$ is increasing in s_i, we can also write $a_i(s) = a_i(\underline{\phi}_i(s_{-i}), s_{-i}) - \int_{[s_i, \underline{\phi}_i(s_{-i}))} a_i(\mathrm{d}s_i, s_{-i})$. Plugging this expression into $\ell_i(a, s_{-i})$, we obtain

$$\begin{aligned}
&\ell_i(a, s_{-i}) \\
=&a_i(\underline{\phi}_i(s_{-i}), s_{-i}) \int_0^{\underline{\phi}_i(s_{-i})} \Big(\frac{\partial w_i}{\partial a_i}(\underline{\phi}_i(s_{-i}), s) f_i(s_i) - \kappa_i F_i(s_i) - \kappa_i(s_i - \underline{\phi}_i(s_{-i})) f_i(s_i)\Big) \mathrm{d}s_i \\
&- \int_{[0, \underline{\phi}_i(s_{-i}))} \Big[\int_0^{s_i} \Big(\frac{\partial w_i}{\partial a_i}(\underline{\phi}_i(s_{-i}), \tilde{s}) f_i(\tilde{s}_i) - \kappa_i F_i(\tilde{s}_i) - \kappa_i(\tilde{s}_i - \underline{\phi}_i(s_{-i})) f_i(\tilde{s}_i)\Big) \mathrm{d}\tilde{s}_i\Big] a_i(\mathrm{d}s_i, s_{-i}) \\
=&a_i(\underline{\phi}_i(s_{-i}), s_{-i}) \int_0^{\underline{\phi}_i(s_{-i})} \frac{\partial w_i}{\partial a_i}(\underline{\phi}_i(s_{-i}), s) f_i(s_i) \mathrm{d}s_i \\
&- \int_{[0, \underline{\phi}_i(s_{-i}))} \Big[\int_0^{s_i} \frac{\partial w_i}{\partial a_i}(\underline{\phi}_i(s_{-i}), \tilde{s}) f_i(\tilde{s}_i) \mathrm{d}\tilde{s}_i - \kappa_i(s_i - \underline{\phi}_i(s_{-i})) F_i(s_i)\Big] a_i(\mathrm{d}s_i, s_{-i}) \\
=&- \int_{[0, \underline{\phi}_i(s_{-i}))} \Big[\int_0^{s_i} \frac{\partial w_i}{\partial a_i}(\underline{\phi}_i(s_{-i}), \tilde{s}) f_i(\tilde{s}_i) \mathrm{d}\tilde{s}_i - \kappa_i(s_i - \underline{\phi}_i(s_{-i})) F_i(s_i)\Big] a_i(\mathrm{d}s_i, s_{-i}),
\end{aligned} \tag{B.8}$$

where the first equality comes from changing the order of integration. The second equality comes from, for all s_i, $\int_0^{s_i} (\tilde{s}_i - \underline{\phi}_i(s_{-i})) f_i(\tilde{s}_i) \mathrm{d}\tilde{s}_i = (s_i - \underline{\phi}_i(s_{-i})) F_i(s_i) - \int_0^{s_i} F_i(\tilde{s}_i) \mathrm{d}\tilde{s}_i$. The third inequality comes from $\int_0^{\underline{\phi}_i(s_{-i})} \frac{\partial w_i}{\partial a_i}(\underline{\phi}_i(s_{-i}), s_i, s_{-i}) f_i(s_i) \mathrm{d}s_i = 0$ by condition C2. By condition C2 again, we know the term in the square bracket in (B.8) is nonnegative. This implies that $\ell_i(a, s_{-i}) \leq 0$. But notice that $a_i^*(s_i, s_{-i})$ is constant over $s_i \in [0, \underline{\phi}_i(s_{-i})]$. Therefore, $\ell_i(a^*, s_{-i}) = 0$.

Using a similar argument and condition C3, we can also show that $h_i(a, s_{-i}) \leq 0$ and $h_i(a^*, s_{-i}) = 0$. Therefore, we know $\partial\mathcal{L}(a) \leq 0$ for all $a \in \Phi$ and $\partial\mathcal{L}(a^*) = 0$.

Finally, using a similar argument as in the calculation of $\partial\mathcal{L}(a)$ (see footnote 30), we can calculate

$$\lim_{\alpha \to 0} \frac{\mathcal{L}(\alpha a + (1-\alpha)a^*) - \mathcal{L}(a^*)}{\alpha} = \partial\mathcal{L}(a) - \partial\mathcal{L}(a^*) \leq 0.$$

Step 3: a^* solves (B.4).

Suppose not. There exists $a \in \Phi$ such that $\mathcal{L}(a) > \mathcal{L}(a^*)$. By concavity from Step 1, $\mathcal{L}(\alpha a + (1-\alpha)a^*) \geq \alpha\mathcal{L}(a) + (1-\alpha)\mathcal{L}(a^*)$ for all $\alpha \in (0, 1)$. Equivalently, $\frac{\mathcal{L}(\alpha a + (1-\alpha)a^*) - \mathcal{L}(a^*)}{\alpha} \geq \mathcal{L}(a) - \mathcal{L}(a^*)$ for all $\alpha \in (0, 1)$. Letting α go to 0 yields $\lim_{\alpha \to 0} \frac{\mathcal{L}(\alpha a + (1-\alpha)a^*) - \mathcal{L}(a^*)}{\alpha} \geq \mathcal{L}(a) - \mathcal{L}(a^*) > 0$, contradicting Step 2. Therefore, a^* is a solution to (B.4), completing the proof. $\square$

Online Appendix C Proofs for Section 4

Proof of Proposition 2. We first verify that all the conditions needed in Theorem 2 are satisfied. For this, we only verify condition U1. All other conditions are straightforward.

We continue to use notation $\underline{g}_i(x, s_{-i})$ and $\bar{g}_i(x, s_{-i})$ defined in the proof of Lemma 3. Moreover, for notational simplicity, let $\tilde{\lambda}_i = \frac{\lambda_i}{\lambda_0}$ for $i = 1, 2$. Consider $\underline{g}_i(x, s_{-i})$. It is easy to calculate that

$$\begin{aligned}
\frac{\partial \underline{g}_i(x, s_{-i})}{\partial x} &= -2\int_0^x \tilde{\lambda}_i F_i(s_i)\mathrm{d}s_i - 2F_i(x)(x - s_{-i}), \\
\frac{\partial^2 \underline{g}_i(x, s_{-i})}{\partial x^2} &= 2F_i(x)\left[\frac{f_i(x)}{F_i(x)}(s_{-i} - x) - (\tilde{\lambda}_i + 1)\right].
\end{aligned}$$

When $s_{-i} = 0$, $\frac{\partial^2 \underline{g}_i(x,0)}{\partial x^2} < 0$ for $x \in (0, 1]$. Therefore, $\underline{g}_i$ is strictly concave and hence strictly quasi-concave. Assume $s_{-i} > 0$. Let $\theta(x) \equiv \frac{f_i(x)}{F_i(x)}(s_{-i} - x) - (\tilde{\lambda}_i + 1)$. Because $\frac{f_i}{F_i}$ is decreasing by Lemma 16, θ is strictly decreasing over $(0, s_{-i}]$. Because $\lim_{x\downarrow 0} \frac{f_i(x)}{F_i(x)} = +\infty$ by Lemma 16 again, we know $\lim_{x\downarrow 0} \theta(x) = +\infty$. Moreover, because $\theta(s_{-i}) < 0$, we know there exists $x' \in (0, s_{-i})$ such that θ is positive over $(0, x')$ and negative over (x', s_{-i}). Clearly, θ is also negative over $[s_{-i}, 1]$. Therefore, over the interval $(0, 1)$, $\frac{\partial^2 \underline{g}_i(\,\cdot\,, s_{-i})}{\partial x^2}$ single-crosses the x-axis from above, implying that $\underline{g}_i(\,\cdot\,, s_{-i})$ is strictly quasi-concave. We can similarly show that $\bar{g}_i(\,\cdot\,, s_{-i})$ is strictly quasi-concave.

From the proof of Lemma 3, we know that $c_i^*(s_{-i}) = \arg\max_{x\in[0,1]} \underline{g}_i(x, s_{-i})$. Observe that $\frac{\partial \underline{g}_i(0, s_{-i})}{\partial x} = 0$ for all s_{-i}. When $s_{-i} = 0$, the above analysis implies that $\frac{\partial \underline{g}_i(x, s_{-i})}{\partial x} < 0$ for $x > 0$. Therefore, $c_i^*(0) = 0$. When $s_i > 0$, the above analysis implies that $c_i^*(s_{-i}) > 0$ and satisfies the first order condition

$$\frac{\partial \underline{g}_i(c_i^*(s_{-i}), s_{-i})}{\partial x} = -2\int_0^{c_i^*(s_{-i})} \tilde{\lambda}_i F_i(s_i)\mathrm{d}s_i - 2F_i(c_i^*(s_{-i}))(c_i^*(s_{-i}) - s_{-i}) = 0,$$

or equivalently

$$c_i^*(s_{-i}) = s_{-i} - \tilde{\lambda}_i \frac{\int_0^{c_i^*(s_{-i})} F_i(s_i)\mathrm{d}s_i}{F_i(c_i^*(s_{-i}))} < s_{-i}. \tag{C.1}$$

Similarly, we can show that $d_i^*(1) = 1$. When $s_{-i} < 1$, we have $d_i^*(s_{-i}) < 1$ and is determined by

$$d_i^*(s_{-i}) = s_{-i} + \tilde{\lambda}_i \frac{\int_{d_i^*(s_{-i})}^1 (1 - F_i(s_i))\mathrm{d}s_i}{1 - F_i(d_i^*(s_{-i}))} > s_{-i}. \tag{C.2}$$

This completes the proof. $\square$

Propositions 3 and 4 are built on the next two simple lemmas. Lemma C.1 is a technical result about log-concavity. It strengthens some of the results in Lemma 16.

Lemma C.1. *If f_i is log-concave, both $s_i \mapsto \int_0^{s_i} F_i(s_i')\mathrm{d}s_i'$ and $s_i \mapsto \int_{s_i}^1 (1 - F_i(s_i'))\mathrm{d}s_i'$ are strictly log-concave. Therefore, $\frac{F_i(s_i)}{\int_0^{s_i} F_i(s_i')\mathrm{d}s_i'}$ is strictly decreasing and $\frac{1-F_i(s_i)}{\int_{s_i}^1 (1-F_i(s_i'))\mathrm{d}s_i'}$ is strictly increasing.*

Proof. We only show that $s_i \mapsto \int_{s_i}^1 (1 - F(s_i'))\mathrm{d}s_i'$ is strictly log-concave. The other one is similar. Consider any $s_i \in (0, 1)$. By part (i) in Lemma 16, we know there exists $s_i'' \in (s_i, 1)$ such that

$$\frac{f_i(s_i)}{1 - F_i(s_i)} \leq \frac{f_i(s_i')}{1 - F_i(s_i')}, \ \forall s_i' \in (s_i, 1),$$

with strictly inequality when $s_i' \in (s_i'', 1)$. This implies

$$\frac{f_i(s_i)}{1 - F_i(s_i)} \int_{s_i}^1 (1 - F_i(s_i'))\mathrm{d}s_i' < \int_{s_i}^1 \frac{f_i(s_i')}{1 - F_i(s_i')}(1 - F_i(s_i'))\mathrm{d}s_i' = 1 - F_i(s_i),$$

which in turn implies

$$\left[\log \int_{s_i}^1 (1 - F_i(s_i'))\mathrm{d}s_i'\right]'' = \frac{f_i(s_i) \int_{s_i}^1 (1 - F_i(s_i'))\mathrm{d}s_i' - (1 - F_i(s_i))^2}{\left(\int_{s_i}^1 (1 - F_i(s_i'))\mathrm{d}s_i'\right)^2} < 0.$$

Therefore, $\int_{s_i}^1 (1 - F_i(s_i'))\mathrm{d}s_i'$ is strictly log-concave. □

Lemma C.2 below shows the monotone comparative statics of agents' unilaterally constrained delegation rules with respect to the parameters. Denote by $(c^*_{i,\lambda_0,\lambda_i}, d^*_{i,\lambda_0,\lambda_i})$ the unilaterally constrained delegation rule for agent i when the importance of coordination is λ_0 and that of his adaptation is λ_i.[31]

Lemma C.2. *For any $s_{-i} \in (0, 1)$, $c^*_{i,\lambda_0,\lambda_i}(s_{-i})$ is strictly increasing in λ_0 and strictly decreasing in λ_i; $d^*_{i,\lambda_0,\lambda_i}(s_{-i})$ is strictly decreasing in λ_0 and strictly increasing in λ_i.*

Proof of Lemma C.2. For example, assume $\bar{\lambda}_i > \underline{\lambda}_i$. Pick any $s_{-i} \in (0, 1)$. For notational simplicity, let $\underline{c} = c^*_{i,\lambda_0,\underline{\lambda}_i}(s_{-i})$ and $\bar{c} = c^*_{i,\lambda_0,\bar{\lambda}_i}(s_{-i})$. By (C.1), we have

$$\underline{c} + \frac{\underline{\lambda}_i}{\lambda_0} \frac{\int_0^{\underline{c}} F_i(s_i)\mathrm{d}s_i}{F_i(\underline{c})} = \bar{c} + \frac{\bar{\lambda}_i}{\lambda_0} \frac{\int_0^{\bar{c}} F_i(s_i)\mathrm{d}s_i}{F_i(\bar{c})} > \bar{c} + \frac{\underline{\lambda}_i}{\lambda_0} \frac{\int_0^{\bar{c}} F_i(s_i)\mathrm{d}s_i}{F_i(\bar{c})}.$$

[31] The unilaterally constrained delegation rule for agent i does not depend on the importance of agent $-i$'s adaptation.

Because $c \mapsto c + \frac{\lambda_i}{\lambda_0}\frac{\int_0^c F_i(s_i)\mathrm{d}s_i}{F_i(c)}$ is strictly increasing by Lemma C.1, we know $\underline{c} > \bar{c}$. This proves that $c^*_{i,\lambda_0,\lambda_i}(s_{-i})$ is strictly decreasing in λ_i. The same argument can be applied to show that $c^*_{i,\lambda_0,\lambda_i}(s_{-i})$ is strictly increasing in λ_0. The proof for $d^*_{i,\lambda_0,\lambda_i}$ is analogous. □

Proof of Proposition 3. Let $(\phi^*_{1,\lambda_0}, \phi^*_{2,\lambda_0})$ be the principal's optimal contingent delegation when the importance of coordination to her is λ_0. For any s_{-i}, We show that $\underline{\phi}^*_{i,\lambda_0}(s_{-i})$ is increasing while $\bar{\phi}^*_{i,\lambda_0}(s_{-i})$ is decreasing in λ_0, for both $i = 1, 2$. For notational simplicity, we suppress λ_i from the previous notation $c^*_{i,\lambda_0,\lambda_i}$ and $d^*_{i,\lambda_0,\lambda_i}$, and directly write c^*_{i,λ_0} and d^*_{i,λ_0}.

Consider $0 < \underline{\lambda}_0 < \bar{\lambda}_0 < \infty$. We show $\bar{\phi}^*_{1,\bar{\lambda}_0} \leq \bar{\phi}^*_{1,\underline{\lambda}_0}$ and $\underline{\phi}^*_{2,\bar{\lambda}_0} \geq \underline{\phi}^*_{2,\underline{\lambda}_0}$. The proof is most easily understood by looking at Figure C.1. Let $(\bar{L}_{1,\lambda_0}, \underline{H}_{2,\lambda_0})$ be the intersection of d^*_{1,λ_0} and c^*_{2,λ_0} for $\lambda_0 \in \{\underline{\lambda}_0, \bar{\lambda}_0\}$. By Lemma C.2, we know $d^*_{1,\bar{\lambda}_0} \leq d^*_{1,\underline{\lambda}_0}$ and $c^*_{2,\bar{\lambda}_0} \geq c^*_{2,\underline{\lambda}_0}$. Hence in Figure C.1, $(\bar{L}_{1,\underline{\lambda}_0}, \underline{H}_{2,\underline{\lambda}_0})$ can only appear in one of the regions i, i, or iii .

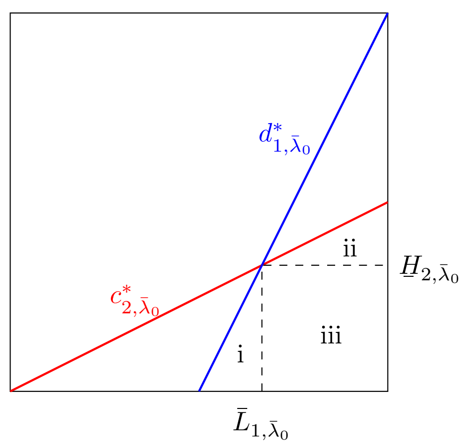


Figure C.1: Graph for the proof of Proposition 3

We claim that, in fact, $(\bar{L}_{1,\underline{\lambda}_0}, \underline{H}_{2,\underline{\lambda}_0})$ can only be in region iii. To see this, note that $c^*_{2,\lambda_0}(d^*_{1,\lambda_0}(\underline{H}_{2,\lambda_0})) = \underline{H}_{2,\lambda_0}$, for $\lambda_0 \in \{\underline{\lambda}_0, \bar{\lambda}_0\}$. Using (C.1), (C.2), and the fact $d^*_{1,\lambda_0}(\underline{H}_{2,\lambda_0}) = \underline{L}_{1,\lambda_0}$, we know

$$0 = \frac{\lambda_2}{\underline{\lambda}_0}\frac{\int_{\bar{L}_{1,\underline{\lambda}_0}}^1 (1 - F_1(s_1))\mathrm{d}s_1}{1 - F_1(\bar{L}_{1,\underline{\lambda}_0})} - \frac{\lambda_1}{\underline{\lambda}_0}\frac{\int_0^{\underline{H}_{2,\underline{\lambda}_0}} F_2(s_2)\mathrm{d}s_2}{F_2(\underline{H}_{2,\underline{\lambda}_0})}$$

$$= \frac{\lambda_2}{\bar{\lambda}_0}\frac{\int_{\bar{L}_{1,\bar{\lambda}_0}}^1 (1 - F_1(s_1))\mathrm{d}s_1}{1 - F_1(\bar{L}_{1,\bar{\lambda}_0})} - \frac{\lambda_1}{\bar{\lambda}_0}\frac{\int_0^{\underline{H}_{2,\bar{\lambda}_0}} F_2(s_2)\mathrm{d}s_2}{F_2(\underline{H}_{2,\bar{\lambda}_0})}.$$

Because $x \mapsto \frac{\int_x^1 (1-F_1(s_1))\mathrm{d}s_1}{1-F_1(x)}$ is strictly decreasing and $x \mapsto \frac{\int_0^x F_2(s_2)\mathrm{d}s_2}{F_2(x)}$ is strictly increasing by Lemma C.1, it is easy to see from the above equation that we can have neither $\bar{L}_{1,\underline{\lambda}_0} \leq \bar{L}_{1,\bar{\lambda}_0}$ and $\underline{H}_{2,\underline{\lambda}_0} < \underline{H}_{2,\bar{\lambda}_0}$, nor $\bar{L}_{1,\underline{\lambda}_0} > \bar{L}_{1,\bar{\lambda}_0}$ and $\underline{H}_{2,\underline{\lambda}_0} \geq \bar{H}_{2,\underline{\lambda}_0}$. In other words, $(\bar{L}_{1,\underline{\lambda}_0},\, \underline{H}_{2,\underline{\lambda}_0})$ can be in neither region i nor region ii.

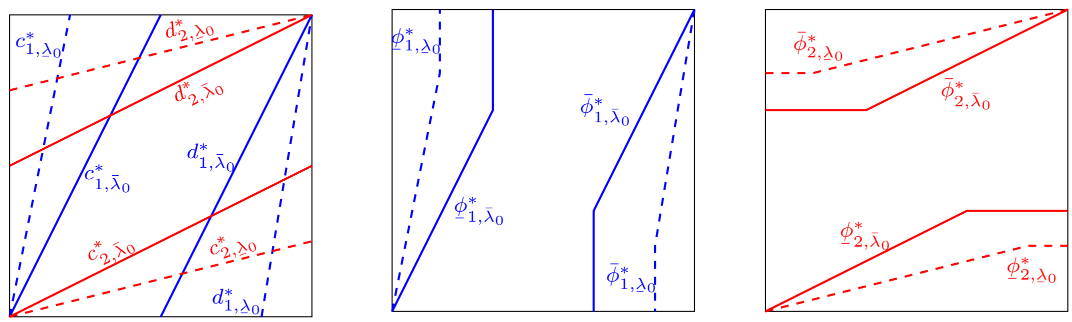
Therefore, $(\bar{L}_{1,\underline{\lambda}_0},\, \underline{H}_{2,\underline{\lambda}_0})$ is in region iii. Equivalently, $\bar{L}_{1,\underline{\lambda}_0} \geq \bar{L}_{1,\bar{\lambda}_0}$ and $\underline{H}_{2,\underline{\lambda}_0} \leq \underline{H}_{2,\bar{\lambda}_0}$. For any $s_2 \in [0,1)$, we then have

$$\bar{\phi}^*_{1,\underline{\lambda}_0}(s_1) = \max\{d^*_{1,\underline{\lambda}_0}(s_1),\, \bar{L}_{1,\underline{\lambda}_0}\} \geq \max\{d^*_{1,\bar{\lambda}_0}(s_1),\, \bar{L}_{1,\bar{\lambda}_0}\} = \bar{\phi}^*_{1,\bar{\lambda}_0}(s_1).$$

Similarly, for any $s_1 \in (0,1]$, we have

$$\underline{\phi}^*_{2,\underline{\lambda}_0}(s_2) = \min\{c^*_{2,\underline{\lambda}_0}(s_2),\, \underline{H}_{2,\underline{\lambda}_0}\} \leq \min\{c^*_{2,\bar{\lambda}_0}(s_2),\, \underline{H}_{2,\bar{\lambda}_0}\} = \underline{\phi}^*_{2,\bar{\lambda}_0}(s_2).$$

Figure C.2 gives an illustration. □



(a) $(c^*_i,\, d^*_i)$ (b) ϕ^*_1 (c) ϕ^*_2

Figure C.2: Importance of coordination and optimal discretion: $\bar{\lambda}_0 > \underline{\lambda}_0$

Proof of Proposition 4. It is a direct implication of Lemma C.2. See Figure C.3 for an illustration. □

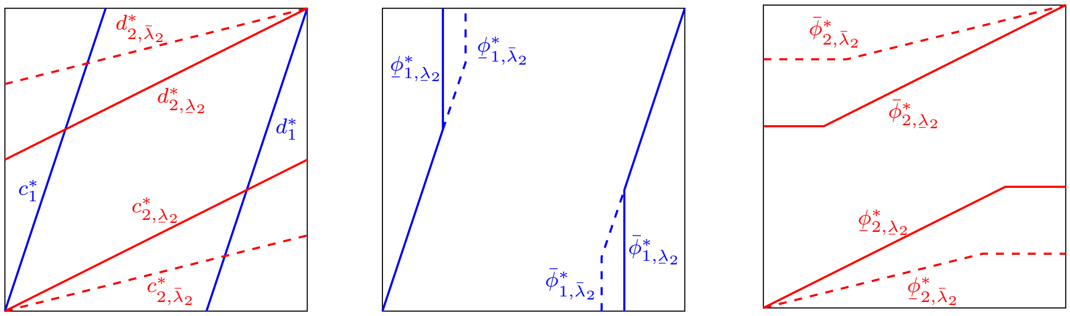
Proposition 5 is a direct implication of Lemma C.3 below. Denote by $(c^*_{i,f_i},\, d^*_{i,f_i})$ i's unilaterally coordinated delegation rule when his state distribution is f_i.

Lemma C.3. *Suppose $0 < \lambda_i < \infty$. Consider two densities $\underline{f}_i$ and $\bar{f}_i$ of agent i's state distribution. If the likelihood ratio $\bar{f}_i/\underline{f}_i$ is (strictly) increasing, then $c^*_{i,\bar{f}_i}(s_{-i}) \geq (>)\, c^*_{i,\underline{f}_i}(s_{-i})$ and $d^*_{i,\bar{f}_i}(s_{-i}) \geq (>)\, d^*_{i,\underline{f}_i}(s_{-i})$ for all $s_{-i} \in (0,1)$.*

Proof of Lemma C.3. Let $\bar{F}_i$ and $\underline{F}_i$ be the c.d.f.'s of $\bar{f}_i$ and $\underline{f}_i$ respectively. Because $\bar{f}_i$ and $\underline{f}_i$ satisfy the (strict) MLRP, we know that, for all $c, d \in (0,1)$,[32]

$$\frac{\int_0^c \bar{F}_i(s_i)\mathrm{d}s_i}{\bar{F}_i(c)} \leq (<) \frac{\int_0^c \underline{F}_i(s_i)\mathrm{d}s_i}{\underline{F}_i(c)} \text{ and } \frac{\int_d^1 (1-\bar{F}_i(s_i))\mathrm{d}s_i}{1-\bar{F}_i(s_i)} \geq (>) \frac{\int_d^1 (1-\underline{F}_i(s_i))\mathrm{d}s_i}{1-\underline{F}_i(s_i)}.$$

[32]See, for example, Theorem 1.C.1 in Shaked and Shanthikumar (2007).



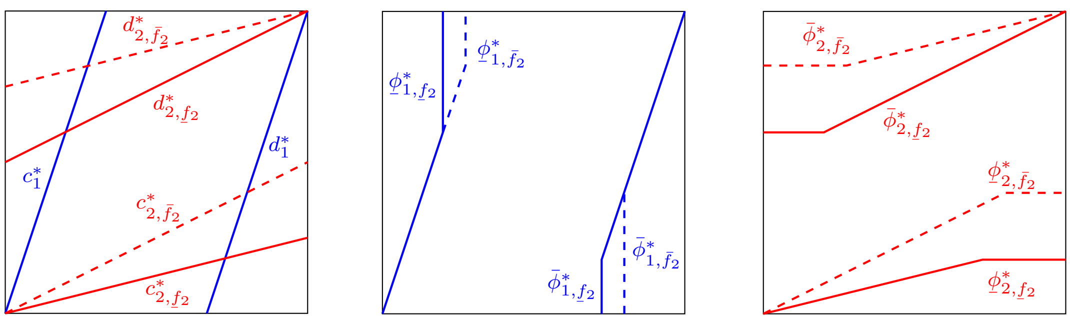
(a) $(c_i^*,\, d_i^*)$ (b) ϕ_1^* (c) ϕ_2^*

Figure C.3: Relative importance and optimal discretion: $\bar{\lambda}_2 > \underline{\lambda}_2$

Consider $s_{-i} \in (0, 1)$. Let $\underline{c} = c^*_{i,\underline{f}_i}(s_{-i})$ and $\bar{c} = c^*_{i,\bar{f}_i}(s_{-i})$. By (C.1), we have

$$\underline{c} + \frac{\lambda_i}{\lambda_0} \frac{\int_0^{\underline{c}} \underline{F}_i(s_i)\mathrm{d}s_i}{\underline{F}_i(\underline{c})} = \bar{c} + \frac{\lambda_i}{\lambda_0} \frac{\int_0^{\bar{c}} \bar{F}_i(s_i)\mathrm{d}s_i}{\bar{F}_i(\bar{c})} \leq (<)\, \bar{c} + \frac{\lambda_i}{\lambda_0} \frac{\int_0^{\bar{c}} \underline{F}_i(s_i)\mathrm{d}s_i}{\underline{F}_i(\bar{c})}.$$

Again, because $c \mapsto c + \frac{\lambda_i}{\lambda_0} \frac{\int_0^c \underline{F}_i(s_i)\mathrm{d}s_i}{\underline{F}_i(c)}$ is strictly increasing, we know $\underline{c} \leq (<)\, \bar{c}$. Figure C.4 provides an illustration. □



(a) $(c_i^*,\, d_i^*)$ (b) ϕ_1^* (c) ϕ_2^*

Figure C.4: State distribution and optimal discretion: $\bar{f}_2/\underline{f}_2$ is increasing