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N-Player War of Attrition with Complete Information

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Abstract

This paper considers a standard war of attrition game in which N heterogeneous players complete for $N - K$ prizes. Each player can win at most one prize. When $K = 1$ (the base model), the game may have an equilibrium in which the strategies follow exponential distributions. The equilibrium surely exists when $N = 2$, but may not exist when $N \geqslant 3$. In the equilibrium, a weaker player behaves 'tougher' and is more likely to win. All players receive an expected payoff of zero. The equilibrium, if exists, is the unique equilibrium in which the strategies follow atomless distributions and have supports of the entire time horizon. The game also has many partially degenerate equilibria, in which the game ends immediately with a probability. When $K \geqslant 2$, there may exist nondegenerate equilibria in which $K-1$ players exit immediately. The model can be extended to cases where winners' payoffs depend on which players exit, or where players face randomly arriving 'defeats'. The findings can be applied to an all-pay auction with ascending bids and complete information.

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1 Introduction

A war of attrition occurs when a group of people 'battle' for a limited number of prizes. Prizes are awarded only after some of the participants give in. The theories of war of attrition have been applied in many areas, including the provision of public goods (Bliss and Nalebuff, 1984; Kambe, 2019), natural oligopolies (Fudenberg and Tirole, 1986; Bulow and Klemperer, 1999; Levin and Peck, 2003), committee voting (Dion et al. 2016; Meyer-ter-Vehn, Smith, and Bognar, 2018; Kwiek, Marreiros, and Vlassopoulos, 2019), bargaining (Abreu and Gul, 2000; Dixit and Olson, 2000), and territorial disputes (Bar-Siman-Tov, 1980; Sorby, 2017).

Although there is a large body of literature in the area, war of attrition games with more than two players and complete information remain understudied. This paper hopes to fill this gap. I consider a standard war of attrition game in which N heterogeneous players compete for $N-K$ prizes. The game ends when K players exit and the other $N-K$ players each win a prize. I first consider the case with $K=1$, called the 'base model', and then consider that with $2 \leq K \leq N-1$. An intuitive approach is adopted to find a strictly mixed strategy equilibrium for the game. The model allows studying problems that cannot arise when there are only two players, for example, $K \geq 2$ or players' valuations of prizes may depend on which other players exit.

Much of the literature on war of attrition with complete information has focused on two-player games. Smith (1974, 213) describes an 'evolutionary stable strategy (ESS)' equilibrium for a symmetric war of attrition game between two fighting animals. The ESS strategy follows an exponential distribution. Bishop and Cannings (1978) extend Smith's model to one that allows for more general rewards and cost functions. Restrictions on the length of the contest are also permitted. Their symmetric two-player game has no ESS equilibrium or a unique ESS equilibrium. Hendricks, Weiss, and Wilson (1988) consider a more general two-player game with complete information, allowing for asymmetric players. The conditions for different types of equilibria are described. Due to the exceptional generality, the 'nondegenerate equilibrium' of the game is not explicitly characterized, which makes it less convenient to observe the implications of the equilibrium.

Some recent papers examine the applications of the theory of war of attrition under complete information. Gul and Pesendorfer (2012) consider a 'war of information', in which two parties with opposing interests provide costly information to voters who choose a policy. The flow of information is continuous and stops when both sides withdraw. The parties are uncertain about the voters' utility from the proposed policy, but they observe a signal, a Brownian motion with a state-dependent drift. They find that increasing one party's cost actually makes that party provide more information and therefore that party is more likely to win. Georgiadis, Kim, and Kwon (2022) consider a war of attrition between two oligopolists under complete information. Time is continuous and values are discounted exponentially. They show that if the players' payoffs whilst in 'war' vary stochastically and their exit payoffs are heterogeneous, the game admits Markov Perfect equilibria in pure strategies only.

A war of attrition may be similar to an all-pay auction. Krishna and Morgan (1996, 344, footnote 2) suggest that a war of attrition can be described as a 'second-price all-pay auction' and an all-pay auction as a 'first-price all-pay auction'. Baye, Kovenock, and de Vries (1996) study a first-price all-pay auction with complete information, in which n bidders bid for one object. The game has no pure strategy equilibrium but has many mixed strategy equilibria. The findings of our model can be applied to an all-pay auction with ascending bids (or 'all-pay Japanese auction') with complete information.

There is a large literature on war of attrition with incomplete information. Nalebuff and Riley (1985) consider a two-player war of attrition with constant costs per period of staying in the game and private information in valuation. They show that the game has a continuum of asymmetric equilibria in general. Fudenberg and Tirole (1986) analyse the strategic exit of firms in a duopoly, in which values are discounted exponentially. The fixed costs or opportunity costs of the firms are privately known. They show that when each firm's costs can be low enough with some probability that staying in the market forever is a dominant strategy, the model has a unique equilibrium. Bliss and Nalebuff (1984) model an ex-ante symmetric game in which multiple individuals decide when to take the initiative and supply a public good. The value of the public good is discounted exponentially. The costs of supplying the good are private information. The analysis can be viewed as an application of the Revelation Principle in mechanism design and the Revenue Equivalence Theorem in auction theory. Bulow and Klemperer (1999) offer a more general (but still ex-ante symmetric) model in which $N+K$ firms compete for N prizes. The firms' valuations of the prizes are private information. Staying in the game incurs constant costs for the firms. The analyses help to explain how long it takes to form a winning coalition. Solving the model is also facilitated by the Revenue Equivalence Theorem.

There are many other studies of the game with incomplete information. To name just a few, Krishna and Morgan (1996) study the war of attrition and all-pay auction with N bidders and one object, when the bidders' signals are affiliated and symmetrically distributed; Hörner and Sahuguet (2011) study a two-player war of attrition with private valuations and endogenous effort levels; Meyer-ter-Vehn, Smith, and Bognar (2018) consider the 'conversational war of attrition' between two biased jurors; and Chen and Ishida (2021) extend a two-player game with incomplete information by incorporating experimentation, i.e., each player's type may change over time as a result of learning-by-doing.

War of attrition games usually have many equilibria. It may help to choose between many equilibria by assuming that there is a small probability that the players are irrational. Kornhauser, Rubinstein, and Wilson (1989) consider a two-player 'concession game' (with discrete time and alternate moves). An irrational mixed strategy for each player is included in the game. They find that the approach may select a unique equilibrium in which the 'weaker' player exits immediately. A player is said to be 'weaker' either if he is more impatient or if his irrational strategy is to wait (in any period) with higher probability. Abreu and Gul (2000) investigate the influence of bargaining 'postures' on bilateral bargaining outcomes, in which players could be irrationally obstinate. One finding is that the delay and inefficiency disappear as the probability of irrationality goes to zero. Kambe (2019) studies an N-player war of attrition game in which the players could be 'noncompromising'. The paper examines which player is likely to exit and when the war of attrition ends quickly.

The rest of this paper proceeds as follows. Section 2 presents a standard war of attrition model in which $N \geq 2$ players compete for $N-K$ prizes. Subsection 2.1 considers the base model in which $K=1$ and the players' valuations of the prizes are fixed. A mixed strategy equilibrium with memoryless strategies is explicitly characterized and a uniqueness theorem is presented. Subsection 2.2 considers the case with $2 \leq K \leq N-1$. It shows that when the players are about equally strong, the game has nondegenerate equilibria in which $K-1$ of the players exit immediately. Section 3 discusses two extensions of the base model. In Subsection 3.1, the winners' valuations of the prizes depend on which player loses. A closed-form mixed strategy equilibrium with memoryless strategies, if it exists, can still be derived. In Subsection 3.2, the players may be forced out of the game due to randomly arriving 'defeats'. Section 4 concludes the paper.

2 The Model

In a war of attrition with continuous and infinite time horizon, $N \geq 2$ risk-neutral players compete for $N-K$ homogeneous prizes, with $1 \leq K \leq N-1$. When exactly K players exit, each of the other $N-K$ players wins a prize and the game ends. Player i's valuation of the prize is $V_i > 0$. Write $\vec{V} \equiv (V_1, \ldots, V_N)$. Player i's cost of staying in the game is $c_i > 0$ per period. Write $\vec{c} \equiv (c_1, \ldots, c_N)$. The players stop paying the costs once they exit or the game ends. Values are not discounted over time. All information is public. The game is denoted by $(N, N-k, \vec{V}, \vec{c})$. Without loss of generality, suppose $\frac{V_1}{c_1} \geq \cdots \geq \frac{V_N}{c_N}$. I use the ratio $\frac{V_i}{c_i}$ to measure the 'strength' of player i in the game. Hence, player 1 is assumed to be the strongest one, and player N is the weakest one.

The players must decide when to exit. A pure strategy of a player is represented by a time $t \in [0, +\infty)$, at which the player exits conditional on the game not having ended yet. A mixed strategy is represented by a cumulative distribution on $[0,+\infty)$. The player taking the strategy draws a time from the distribution and exits at that time conditional on the game not having ended then. All players simultaneously decide their strategies. As in Hendricks, Weiss, and Wilson (1988), the game considered here is of 'reduced normal form', since the strategies may not fully describe the players' possible plans upon reaching time t. I will consider the 'base model' $(N,N-1,\vec{V},\vec{c})$ first, and then extend the results to $(N,N-K,\vec{V},\vec{c})$.

2.1 N players compete for $N-1$ prizes

The game $(N,N-1,\vec{V},\vec{c})$ can be viewed as a typical public good provision game. It has many degenerate war of attrition equilibria, in which one of the players exits immediately and the others commit to waiting for a sufficiently long time. For example, pure strategy profile $(t_1,\dots,t_N)$ with $t_i = 0$ for an $i \in \{1,\dots,N\}$, and $t_j \geq \frac{V_i}{c_i}$ for all $j \neq i$, is a Nash equilibrium of the game. In the equilibrium, the players other than i 'threaten' to wait for a sufficiently long time, and therefore render immediate exit optimal for player i. The game ends immediately and no attrition occurs. It is possible for a relatively strong player to lose the game in the equilibria.

In a nondegenerate war of attrition equilibria, the players adopt strictly mixed strategies. Since the game is defined on a time horizon, each player should constantly update his beliefs about the other players' future moves as the game proceeds. Only when the players' strategies follow exponential distributions, which are 'memoryless' over time, is updating unnecessary. Since the game $(N,N-1,\vec{V},\vec{c})$ physically remains the same at any point of time before it ends, we may conjecture that the game has an equilibrium with memoryless strategies. I characterize a candidate equilibrium of the game assuming the players adopt memoryless strategies, and then show it is indeed an equilibrium. The cumulative distribution function of an exponential distribution on $[0,+\infty)$ can be written as $F_\lambda(x) = 1-e^{-\lambda x}$, with $\lambda > 0$. The basic properties of the distribution are summarized in following lemma.

Lemma 1. *If independent random variables $\tilde{x}_1$ and $\tilde{x}_2$ follow exponential distributions $F_{\lambda_1}(\cdot)$ and $F_{\lambda_2}(\cdot)$, respectively, then*

(i) $E(\tilde{x}_i) = \frac{1}{\lambda_i}, i \in \{1,2\}$;

(ii) $Prob(\tilde{x}_i > x+y | \tilde{x}_i > x) = Prob(\tilde{x}_i > y), \quad for \quad x,y \geq 0$;

(iii) $Prob(\tilde{x}_i \leq \tilde{x}_j) = \frac{\lambda_i}{\lambda_i+\lambda_j}$, $i,j \in \{1,2\}, \quad i \neq j$;

(iv) *Variable* $\tilde{z} \equiv min\{\tilde{x}_1, \tilde{x}_2\}$ *follows cumulative distribution* $F_{\lambda_1+\lambda_2}(x)$.

Nondegenerate war of attrition equilibrium

For simplicity, I denote a mixed strategy with exponential distribution $F_\lambda(\cdot)$ by the 'rate parameter' λ. A larger λ represents a 'softer' strategy, which means the player is less patient in playing the game and more likely to lose. A mixed strategy profile of the game is then represented by $\vec{\lambda} \equiv (\lambda_1, \ldots, \lambda_N)$. Given the other players' strategies λ_{-i}, player i faces a suppositional 'player' with strategy $\sum\limits_{n=-i} \lambda_n$ (Lemma 1(iv)). His probability of losing the game is $\lambda_i(\sum\limits_{n=1}^{N} \lambda_n)^{-1}$ (Lemma 1(iii)), and the expected duration of the game is $(\sum\limits_{n=-i}^{N} \lambda_n)^{-1}$ (Lemma 1$(i \ \& \ iv)$). All players have the same expected waiting time in the game, regardless of whether they win or lose. Hence, player i's strategy λ_i is the solution of problem

$$\max_{\lambda_i \geq 0} \quad u_i(\lambda_i) = [1 - \lambda_i(\sum_{n=1}^{N} \lambda_n)^{-1}]V_i - (\sum_{n=1}^{N} \lambda_n)^{-1} c_i.$$

The first-order derivative of the objective function is

$$u_i^{'}(\lambda_i) = (\sum_{n=1}^{N} \lambda_n)^{-2}[-(\sum_{n=-i} \lambda_n)V_i + c_i].$$

which implies

$$u_i^{'}(\lambda_i) > 0 \quad \text{if} \quad (\sum_{n=-i} \lambda_n)V_i < c_i,$$

$$u_i^{'}(\lambda_i) = 0 \quad \text{if} \quad (\sum_{n=-i} \lambda_n)V_i = c_i,$$

$$u_i'(\lambda_i) < 0 \quad \text{if} \quad (\sum_{n=-i} \lambda_n) V_i > c_i,$$

In words, $u_i(\lambda_i)$ is strictly increasing when $(\sum_{n=-i} \lambda_n) V_i < c_i$, and is strictly decreasing when $(\sum_{n=-i} \lambda_n) V_i > c_i$, It is constant when $(\sum_{n=-i} \lambda_n) V_i = c_i$. Hence player i's 'reaction function' $\lambda_i(\lambda_{-i})$ is

$$\lambda_i(\lambda_{-i}) \begin{cases} = +\infty & \text{if} \quad (\sum_{n=-i} \lambda_n) V_i < c_i, \\ \in [0, +\infty) & \text{if} \quad (\sum_{n=-i} \lambda_n) V_i = c_i, \quad i = 1, \ldots, N \\ = 0 & \text{if} \quad (\sum_{n=-i} \lambda_n) V_i > c_i. \end{cases}$$

Hence, a nondegenerate war of attrition equilibrium with memoryless strategies is possible only when $(\sum_{n=-i} \lambda_n) V_i = c_i$, which implies

$$\sum_{n=1}^{N} \lambda_n = \frac{1}{N-1} \sum_{n=1}^{N} \frac{c_n}{V_n} \quad \textit{and} \quad \lambda_i^* = \frac{1}{N-1} (\sum_{n=1}^{N} \frac{c_n}{V_n}) - \frac{c_i}{V_i}, \quad i = 1, \ldots, N.$$

Given the suppositional player –i's mixed strategy $\sum_{n=-i} \lambda_n$, player i is indifferent to any exponential mixed strategy. The following proposition shows that player i is also indifferent to any pure strategy $t \in [0, +\infty)$.

Proposition 1. *The game $(N, N-1, \vec{V}, \vec{c})$ has a nondegenerate mixed strategy equilibrium $\vec{\lambda}^* \equiv (\lambda_1^*, \ldots, \lambda_N^*)$ with*

$$\lambda_i^* = \frac{1}{N-1} (\sum_{n=1}^{N} \frac{c_n}{V_n}) - \frac{c_i}{V_i}, \quad i = 1, \ldots, N.$$

conditional on $\vec{\lambda}_1^* > \vec{0}$. All players obtain zero expected payoffs in the equilibrium.

Proof. Assuming the players adopt mixed strategies with exponential distributions, a nondegenerate mixed strategy equilibrium $\vec{\lambda}$ must satisfy $(\sum_{n=-i} \lambda_n) V_i = c_i, \quad i = 1, \ldots, N$, as shown previously. The candidate equilibrium strategies can be solved from them, which are the $(\lambda_1^*, \ldots, \lambda_N^*)$ stated in the proposition. Since the support of an exponential strategy is $[0, +\infty)$, we need to show that given the other players' strategies $\vec{\lambda}_{-i}^*$, player $i \in \{1, \ldots, N\}$ is indifferent to any pure strategy $t \in [0, +\infty)$.

Indeed, under strategy profile $\vec{\lambda}^*$, player i faces a suppositional 'player $-i$' with strategy $\sum\limits_{n=-i}\lambda_n = \frac{c_i}{V_i}$ (Lemma 1(iv)). Player i's payoff from pure strategy $t \in [0,+\infty)$ is

$$
\begin{aligned}
&\int_0^t (V_i - xc_i)dF_{\frac{c_i}{V_i}}(x) - (1-F_{\frac{c_i}{V_i}}(t))tc_i \\
&= V_iF_{\frac{c_i}{V_i}}(t) - [tc_iF_{\frac{c_i}{V_i}}(t) - c_i\int_0^t F_{\frac{c_i}{V_i}}(x)dx] - [tc_i - tc_iF_{\frac{c_i}{V_i}}(t)] \\
&= V_i(1-e^{-\frac{c_i}{V_i}t}) - tc_iF_{\frac{c_i}{V_i}}(t) + c_i(t + \frac{V_i}{c_i}e^{-\frac{c_i}{V_i}t} - \frac{V_i}{c_i}) - tc_i + tc_iF_{\frac{c_i}{V_i}}(t) \\
&= 0.
\end{aligned}
$$

Hence, $\vec{\lambda}^*$ is a mixed strategy equilibrium of the game and the equilibrium payoffs of the players are all zero. ■

The equilibrium described in Proposition 1 is a subgame perfect equilibrium. Since all strategies are completely memoryless and the game ends immediately when one of the players moves, no player has an incentive to adjust his strategy over the course of the game. More specifically, the strategies truncated at any point in time remain the same strategies because they follow exponential distributions. In contrast, the pure-strategy equilibria mentioned previously may contain non-credible threats.

There is a notable difference between the cases with $N = 2$ and $N \geqslant 3$ in the equilibrium described in Proposition 1. When $N = 2$, $(\lambda_1^*, \lambda_2^*) = (\frac{c_2}{V_2}, \frac{c_1}{V_1}) > 0$, which is always a valid mixed strategy equilibrium. But when $N \geqslant 3$, λ_i^* may be nonpositive when $\frac{c_i}{V_i}$ is too large. In other words, the equilibrium exists only when the weakest players are not too weak.

Proposition 1 demonstrates a value-destroying mechanism in strategic interaction with complete information. Although the players have different strengths, they all receive expected payoffs of zero. This finding defies the common sense that a stronger player should be in a more favourable position in a 'battle'. It should be noted that the war of attrition game considered here is different from Benoit's (1984) 'deep pocket' game, in which a financially strapped player may quickly succumb because he expects to lose eventually. The following corollary is straightforward from proposition 1.

Corollary 1. *In the nondegenerate mixed strategy equilibrium* $\vec{\lambda}^*$ *of the game* $(N, N-1, \vec{V}, \vec{c})$,

(i) $\lambda_1^* \geq \ldots \lambda_N^*$, *i.e., a weaker player is 'tougher' in playing the game;*

(ii) *The probability for player i to win the game is*

$$1 - \frac{\lambda_i^*}{\sum_{n=1}^{N} \lambda_n^*} = (N-1)(\sum_{n=1}^{N} \frac{c_n}{V_n})^{-1} \frac{c_i}{V_i}, \quad i \in \{1, \ldots, N\}.$$

Hence, a weaker player is more likely to win the war of attrition.

(iii) *The duration of the game, denoted* T^**, follows an exponential distribution with rate*

$$\lambda^* \equiv \sum_{n=1}^{N} \lambda_n^* = \frac{1}{N-1}(\sum_{n=1}^{N} \frac{c_n}{V_n}).$$

The expected duration of the game $E(T^*) = \frac{1}{\lambda^*}$.

Corollary $1(i)$ can be understood intuitively. Given the equilibrium strategies of the other players, a player must be indifferent to exit now or Δt later. The player balances the additional cost of remaining in the game against the expected benefit of winning the game in the Δt period. As the cost of remaining in the game is relatively high for a weaker player, the probability of winning the game in the Δt period must also be high. But the latter is only possible if the other (relatively stronger) players behave softer in the game. Therefore, a stronger player must choose a softer strategy in the nondegenerate equilibrium, and vice versa.

The counter-intuitive outcome can also be found in the literature. In Dixit and Shapiro's (1986) market entry game with perfect information and discrete rounds, firms that have not yet entered a market decide whether to enter at each round. Levin and Peck (2003) show that Dixit and Shapiro's (1986) model typically has firms with higher entry costs mix with a higher probability of entry than those with lower entry costs (Proposition 8 in Levin and Peck (2003, 547)).

The example below shows that in the market exit game of a 'natural oligopoly,' a more efficient firm may be more likely to exit an unprofitable industry. It also illustrates the possibility that the equilibrium presented in Proposition 1 may not exist when there are more than two players.

Example 1. There are three firms in a market for a homogeneous product. The firms have the same marginal cost, but different fixed costs (per period). After a permanent negative demand shock, the market can only accommodate two firms, i.e., one of them must exit the market before the other two firms regain profitability. A war of attrition is invoked to determine which firm exits. As they have different fixed costs, these firms lose money at different rates before the war ends.

Specifically, let the firms' marginal costs be zero and fixed costs per period be $f_1 = 39$, $f_2 = 40$, and $f_3 = 41$, respectively. The market demand is $P = 24 - Q$ per period after the demand shock. The firms compete by simultaneously choosing output quantities, conditional on staying in the market. If all three firms are in the market, one can check that the equilibrium price is 6 and each firm's gross profit per period is $\tilde{\pi}^3 = 36 < 39$. If only two firms are in the market, the equilibrium price is 8 and each firm's gross profit per period is $\tilde{\pi}^2 = 64 > 41$.

The firms' net losses per period are $(c_1, c_2, c_3) = (3, 4, 5)$ during the war of attrition. Conditional on winning, the firms' net profits are $(\pi_1, \pi_2, \pi_3) = (25, 24, 23)$ per period after the war. Assume that the winning firms' discounted permanent profits are proportional to the net profits per period. The 'strengths' of the firms in the war of attrition can be represented by $(\frac{\pi_1}{c_1}, \frac{\pi_2}{c_2}, \frac{\pi_3}{c_3}) = (8.33, 6, 4.6)$. The most efficient firm, 1, is the strongest one in the game. According to Proposition 1, the nondegenerate equilibrium mixed-strategies prescribed in Proposition 1 are

$$(\lambda_1^*, \lambda_2^*, \lambda_3^*) = (0.132, 0.085, 0.035).$$

The firms' probabilities of exiting the market are

$$\frac{\lambda_1^*}{\lambda_1^*+\lambda_2^*+\lambda_3^*}=0.524, \quad \frac{\lambda_2^*}{\lambda_1^*+\lambda_2^*+\lambda_3^*}=0.339, \quad \text{and} \quad \frac{\lambda_3^*}{\lambda_1^*+\lambda_2^*+\lambda_3^*}=0.137,$$

respectively. Hence, the most efficient firm 1 has the highest probability of exiting the market in the war of attrition.

One can check that the consumer surplus during the war of attrition is 162 per period, and the social welfare is $162-3-4-5=150$ per period. The 'post-war' consumer surplus is 128 per period, and the social welfare may be $128+24+23=175$, $128+25+23=176$, or $128+25+24=177$ per period, depending on which firm exits. From the social welfare perspective, the first-best outcome is for firm 3 to exit immediately, thus achieving social welfare of 177 per period immediately. The war of attrition not only creates a 'war period' with low social welfare (which is 150), but also tends to create a less efficient 'post-war' outcome.

If the firms' fixed costs were $f_1=61$, $f_2=62$, and $f_3=63$ respectively, the firms would lose $(c_1,c_2,c_3)=(25,26,27)$ per period during the war of attrition, and earn net profits per period $(\pi_1,\pi_2,\pi_3)=(3,2,1)$ after the war. Therefore,

$$(\lambda_1^*,\lambda_2^*,\lambda_3^*)=(15.83,11.17,-2.833).$$

It cannot be an equilibrium strategy profile because $\lambda_3^*<0$. In this case, the game does not have an equilibrium as described in Proposition 1. ■

In a war of attrition with private valuations, a player typically chooses a tougher strategy when his (privately known) valuation is higher (Bliss and Nalebuff, 1984; Bulow and Klemperer, 1999). The finding does not conflict with ours that a player chooses a softer strategy when his (publicly known) valuation is higher. The two findings tell different stories. The first one is about how a player's strategy 'vertically' depends on his own possible type, whilst the second is about how a player's strategy 'horizontally' depends on his relative strength compared to other players. In a war of attrition with incomplete information, it is possible for a stronger player to behave softer. The following example illustrates this possibility.

Example 2. Consider a war of attrition in which three players compete for two prizes. They all value the prize at V. Player 1's cost per period $c_1 \in \{c_1^L, c_1^H\}$, $c_1^L < c_1^H$, is his private information, with

$$Prob(c_1 = c_1^L) = 1 - p \quad \text{and} \quad Prob(c_1 = c_1^H) = p \in (0,1).$$

Players 2 and 3's costs per period are c_2 and c_3, respectively, which are publicly known.

The game has a Perfect Bayesian Nash Equilibrium with exponential mixed strategies $(\lambda_1(c_1^L), \lambda_1(c_1^H), \lambda_2, \lambda_3)$, in which

$$\lambda_1(c_1^L) = \frac{1}{2V}(c_2 + c_3 - c_1^L), \quad \lambda_1(c_1^H) = +\infty,$$

$$\lambda_2 = \frac{1}{2V}(c_1^L + c_3 - c_2), \quad \text{and} \quad \lambda_3 = \frac{1}{2V}(c_1^L + c_2 - c_3)$$

If player 1 is the high-cost type, he exits immediately, which happens with probability p. Otherwise, all players adopt the strategy specified in Proposition 1. Note that if player 1 is the high-cost type, the strategies of players 2 and 3 induce player 1 to exit immediately. In equilibrium, player 1 receives an expected payoff of zero. Players 2 and 3 each receive a positive expected payoff pV. ■

Example 2 has some interesting implications. First, when $c_1^L < c_1^H < c_2$ (or $c_1^L < c_1^H < c_3$), player 1 is certainly stronger than player 2 (or 3). But in the proposed equilibrium, player 1 either exits immediately or adopts a softer strategy $\lambda_1(c_1^L) > \lambda_2$ (or $\lambda_1(c_1^L) > \lambda_3$). Second, private information may not benefit the players who own it. On the contrary, it may benefit those who do not own it. A prerequisite, of course, is that the information asymmetry itself is common knowledge. Third, the presence of private information may weaken the war of attrition and lead to an increase in the total surplus of the players. The nondegenerate war of attrition equilibrium described in Proposition 1 represents a 'perfect' mechanism that eliminates all possible surpluses in a strategic interaction. Even information asymmetries can thwart the 'perfection' and raise the total surplus.

Corollary $1(iii)$ implies that the war of attrition lasts longer when the players become stronger, which is reasonable as it takes more time to deplete more potential surpluses. Another related issue is how the expected duration of the game varies as the number of players increases. When there are more players, there is a stronger incentive to wait for others to drop out, which tends to prolong the duration of the game. However, as the game can be ended by more players, the duration of the game may also be shortened. Corollary $1(iii)$ shows that if we fix the 'harmonic mean' $H \equiv N/(\sum_{n=1}^{N} \frac{c_n}{V_n})$ of the players' strengths, the expected duration of the game, $(1-\frac{1}{N})H$, increases with the number of players and approaches the harmonic mean.

Partially degenerate equilibria

The game $(N, N-1, \vec{V}, \vec{c})$ has many equilibria in which the nondegenerate war of attrition only occurs with a probability. Suppose the equilibrium $\vec{\lambda^*} = (\lambda_1^*, \ldots, \lambda_N^*)$ described in Proposition 1 exists. Consider the following strategy profile. One of the players, say $i \in \{1, \ldots, N\}$, exits immediately with probability $p \in (0,1)$ and chooses the exponential strategy λ_i^* with probability $1-p$. All other players, i.e., $j \neq i$, choose the exponential strategy λ_j^* respectively. We can check that the strategies also constitute a Nash equilibrium. Player i receives a zero expected payoff and player $j \neq i$ receives a positive expected payoff pV_j in the equilibrium. I call them 'partially degenerate equilibria'.

Similar to a finding of Hendricks, Weiss, and Wilson (1988), any 'accessible' mass point in the strategies of an equilibrium of the game $(N, N-K, \vec{V}, \vec{c})$ must be at $t = 0$. 'Accessible' here means that the point in time could be reached during the course of the game. If a mass point only serves as a threat, it is not accessible in equilibrium. Suppose $t \in (0, +\infty)$ is an accessible mass point in the support of player i's mixed strategy. No other players would exit in interval $[t-\varepsilon, t]$, with $\varepsilon > 0$ small enough, because it is worthwhile to wait for the possible exit of player i at t. Since no other player would exit in $[t-\varepsilon, t]$, player i can do strictly better by moving the mass point from t to $t-\varepsilon$. Hence, $t>0$ cannot be an accessible mass point in player i's mixed strategy.

Moreover, in the game $(N, N-1, \vec{V}, \vec{c})$, there cannot be two or more players who exit with strictly positive probability at $t=0$. Otherwise, one of them has an incentive to hold on a second so that he can take advantage of the possible exit of the others. Similarly, we can show that in the game $(N, N-K, \vec{V}, \vec{c})$, it is not possible to have $K+1$ or more players exit with strictly positive probabilities at $t=0$. I summarize the two results mentioned above in the following lemma.

Lemma 2. *In the war of attrition game $(N, N-K, \vec{V}, \vec{c})$, any accessible mass point in equilibrium strategies must be at $t=0$. And, it is impossible to have $K+1$ or more players exit with strictly positive probabilities at $t=0$.*

Uniqueness

It can be shown that if the mixed strategy equilibrium described in Proposition 1 exists, then it is the only equilibrium provided that the players' strategies follow atomless distributions over the entire time horizon.

Proposition 2. *If atomless cumulative distributions $G_1, \ldots, G_N$ with supports of $[0, +\infty)$ constitute a mixed strategy equilibrium of the game $(N, N-1, \vec{V}, \vec{c})$, then*

$$G_i(x) = 1 - e^{-[\frac{1}{N-1}(\sum_{n=1}^{N} \frac{c_n}{V_n}) - \frac{c_i}{V_i}]x} = F_{\lambda_i^*}(x), \quad i = 1, \ldots, N.$$

Proof. With the equilibrium strategy profile $(G_1, \ldots, G_N)$, given the others' strategies, player $i \in \{1, \ldots, N\}$

faces a suppositional 'rival' whose strategy is represented by distribution

$$G_{-i}(x) = 1 - \prod_{n=-i}(1-G_n(x)) = 1 - \frac{\prod_{n=1}^{N}(1-G_n(x))}{1-G_i(x)}, \quad x \in [0,+\infty).$$

Player i should be indifferent to any pure strategy $t \in [0,+\infty)$, i.e.,

$$\int_0^t (V_i - c_i x) dG_{-i}(x) - (1-G_{-i}(t))tc_i = C, \quad t \in [0,+\infty)$$

where C stands for a constant. Differentiating both sides of the equation with respect to t, it becomes

$$G'_{-i}(t)V_i - [1-G_{-i}(t)]c_i = 0, \quad \text{for} \quad t \in [0,+\infty), \quad i.e.,$$

$$\frac{(1-G_{-i}(t))'}{(1-G_{-i}(t))} = -\frac{c_i}{V_i}, \quad \text{for} \quad t \in [0,+\infty).$$

Since the distributions are atomless, we have $G_{-i}(0) = 0$. The differential equation implies

$$G_i(x) = 1 - e^{-\frac{c_i}{V_i}x}, \quad \text{for} \quad x \in A_i,$$

i.e., the suppositional player $-i$ must adopt an exponential strategy with rate $\frac{c_i}{V_i}$. Hence,

$$1 - \frac{\prod_{n=1}^{N}(1-G_n(x))}{1-G_i(x)} = 1 - e^{-\frac{c_i}{V_i}x}, \quad i = 1,\ldots,N.$$

From the N equations above, we have

$$G_i(x) = 1 - e^{-[\frac{1}{N-1}(\sum_{n=1}^{N}\frac{c_n}{V_n}) - \frac{c_i}{V_i}]x} = F_{\lambda_i^*}(x), \quad i = 1,\ldots,N.$$

Therefore, all players must adopt mixed strategies with exponential distributions, and the equilibrium strategy profile is exactly the $(\lambda_1^*,\ldots,\lambda_N^*)$ given in Proposition 1. ■

Proposition 2 can be viewed as a generalisation of Smith's (1974, 214) analysis on the ESS equilibrium. It establishes the uniqueness of the nondegenerate war of attrition equilibrium provided that the strategies have atomless distributions and support $[0,+\infty)$. The proposition also offers a direct method for finding the nondegenerate war of attrition equilibrium. This paper proposes a more intuitive way to do this. The idea may be applied to other stationary dynamic games.

2.2 N players compete for $N-K$ prizes

The game $(N, N-K, \vec{V}, \vec{c})$, with $K \geq 2$, also has many pure-strategy equilibria, in which K players exit immediately and the others commit to waiting long enough. With mixed strategies, the game may reach a point at which exactly $N-K+1$ players remain, at which point the subgame fits the base model. We can imagine possible equilibria of the game in which $K-1$ players exit immediately and the remaining $N-K+1$ players choose the mixed strategies described in Proposition 1 or the associated partially degenerate equilibria. For these strategies to constitute a Nash equilibrium, two conditions must be satisfied: first, the subgame of the last $N-K+1$ players has an equilibrium described in Proposition 1; second, among the $K-1$ early quitters, none has an incentive to deviate. If a player is far stronger than all the others, immediate exit may not be his best strategy.

Proposition 3. *In the game $(N, N-k, \vec{V}, \vec{c})$, with $N \geq 3$ and $2 \leq k \leq N-1$, suppose that the $k-1$ earliest quitters do not include all the strongest players, and that the subgame of the remaining $N-k+1$ players have an equilibrium as described by Proposition 1. Then, the game has an equilibrium in which the $k-1$ earliest quitters quit at the beginning of the game.*

Proof. First, after exactly $k-1$ players have exited, at least one of the remaining $N-k+1$ players expect a zero payoff in the nondegenerate or partially degenerate equilibria of the subgame (Proposition 1). If the earliest $k-1$ quitters do not exit at the beginning of the game, then at least one of the remaining $N-k+1$ players will receive a negative expected payoff, which cannot happen in equilibrium. Therefore, the earliest $k-1$ exits must occur at the very beginning of the game.

Second, if one of the earliest quitters deviates to a pure strategy $t > 0$, he wins only if two players with the exponential mixed strategies exit before t. Such a deviation is more profitable for a stronger player.

Since the earliest quitters do not include all the strongest players, in the subgame of $N-K+1$ players, even the strongest player's expected payoff is zero from any pure strategy. Furthermore, waiting for two independent exits takes more expected time (and costs) than waiting for one. Hence, the above deviation cannot be profitable, even for one of the strongest players. ■

Proposition 3 suggests that the game $(N, N-K, \vec{V}, \vec{c})$ may immediately shrink to a game with $N-K+1$ players competing for $N-K$ prizes, which fits the base model. Thus, one can characterize a set of nondegenerate or partially degenerate war of attrition equilibria for the game. The condition that not all the strongest players are among the earliest $K-1$ quitters is a sufficient but not necessary condition for equilibrium. In particular, if all players are almost equally strong, then the $K-1$ early quitters can be randomly chosen from the N players.

Proposition 3 is similar to a conclusion of Bulow and Klemperer (1999). In their (ex-ante symmetric) model, $N+K$ firms with private information on valuation compete for N prizes through a war of attrition. In their 'natural oligarchy' setting, Bulow and Klemperer show that $K-1$ firms with the lowest valuations immediately drop out, leaving $N+1$ firms to battle for the N prizes. In our model, N heterogeneous players compete for $N-K$ prizes under symmetric information. The game may have strictly mixed strategy equilibria in which $K-1$ players immediately drop out, leaving $N-K+1$ players to battle for the $N-K$ prizes. The players who exit immediately are not necessarily those with the lowest valuations (or the 'weakest' players).

The intuition behind the findings is different. Bulow and Klemperer (1999, 181) explain the intuition in terms of marginal analysis. Specifically, when $K>1$ exits are still required for the game to end and a player is within ε of his planned dropout time, the player's cost of waiting as planned is of order ε, but his benefit from winning the game during that period is of order ε^K. So, for small ε he will prefer to exit now rather than wait, which means delay is possible only when one exit is required for the game to end. In our game with the proposed mixed strategy equilibria, when the game reaches the stage in which one more exit is needed to end the game, at least one player who remains in the game expects zero payoff at that point in time. The game must reach this stage immediately; otherwise, there is a player who ends up with a negative expected payoff, which cannot happen in an equilibrium of the game.

All-pay Japanese auction with complete information

Our model can be applied to an all-pay auction with ascending bids, which I call ‘all-pay Japanese auction’. Suppose there are N bidders and $N-K$ objects. The bidders’ valuations of the objects are $V_1,\ldots,V_N$, respectively, with $V_1 \geq \cdots \geq V_N > 0$. Information is complete. In the auction, an auctioneer increases the open bid from zero. A bidder can exit the ongoing auction by paying the current bid without winning anything. Once K bidders exit, the auction ends and each of the other $N-K$ bidders wins an object at the stop price.

A pure strategy of a bidder in the all-pay Japanese auction is represented by an exit price, and a mixed strategy is represented by a cumulative distribution on $[0,+\infty)$. It is easy to see that the social optimal and revenue maximising allocation entails bidders $1,\ldots,N-K$ winning the objects. The corresponding revenue is $\sum_{n=1}^{N-K} V_n$. The auction has many pure strategy equilibria, in which K of the bidders exit immediately.

In the auction game with $K=1$, one can show that a mixed strategy equilibrium with exponential strategies can be solved from

$$(\sum_{n=-i} \lambda_n)V_i = 1, \quad i=1,\ldots,N,$$

which implies

$$\lambda_i^* = \frac{1}{N-1}(\sum_{n=1}^{N} \frac{1}{V_n}) - \frac{1}{V_i}, \quad i=1,\ldots,N.$$

Even if the open bid has exceeded the player’s valuation, a bidder may not give up, as it is better to win a prize at a high price than to lose it at the same price.

A bidder with a lower valuation is more likely to win an object in the auction. All bidders receive an expected surplus of zero. According to Corollary 1(iii), the expected stop bid is $(N-1)/(\sum_{n=1}^{N}\frac{1}{V_n})$. If we denote the 'harmonic mean' of the players' valuations by $\tilde{V} = N/(\sum_{n=1}^{N}\frac{1}{V_n})$, the expected stop bid is $\frac{N-1}{N}\tilde{V}$. Since all bidders must pay the bid, the expected revenue from the auction is $(N-1)\tilde{V}$. As long as the bidders are heterogeneous, the equilibrium outcome is neither Pareto optimal nor revenue maximising, as the good may not be allocated to the bidder with the highest valuation.The auction also has many partially degenerate equilibria where one of the bidders gives up with a positive probability at the start of the auction; otherwise, all bidders follow the mixed strategies mentioned above.

We can apply Proposition 3 to the all-paying Japanese auction in which $K \geq 2$. In addition to the obvious pure strategy equilibria, the auction may have many strictly mixed strategy equilibria, in which $K-1$ bidders exit immediately and the other bidders adopt the nondegenerate or partially degenerate mixed strategies described previously. In particular, if there is only one object for sale, the auction may immediately shrink to a game between two players.

3 Extensions

This section discusses two extensions of the base model. The first one is that a player's valuation of the prize depends on which other player drops out. This can only happen if the game has three or more players. For example, in an exit game in a 'natural oligopoly', the winners' profits may depend on which firms exit. The second extension is that there is some random exogenous event that may cause a player to exit the game early (Asako, 2015). For example, an animal in a fight may be forced to submit due to an unexpected injury. In an exit game, an outside option that appears unexpectedly may induce a firm to quit early.

3.1 Interdependent valuations

In the war of attrition game in which N players compete for $N-1$ prizes, suppose player i's valuation of the prize, conditional on player $j \neq i$, exits and is $V_{ij} > 0$. We define a matrix U by

$$U \equiv \begin{bmatrix} 0 & V_{21} & \dots & V_{(N-1)1} & V_{N1} \\ V_{12} & 0 & \dots & V_{(N-1)2} & V_{N2} \\ \vdots & \vdots & \ddots & \vdots & \\ V_{1(N-1)} & V_{2(N-1)} & \dots & 0 & V_{N(N-1)} \\ V_{1N} & V_{2N} & \dots & V_{(N-1)N} & 0 \end{bmatrix}.$$

The jth row of the matrix gives the players' valuations at the exit of player j. In the base model of the previous section, the valuation matrix degenerates to

$$U = \begin{bmatrix} 0 & V_2 & \dots & V_{N-1} & V_N \\ V_1 & 0 & \dots & V_{N-1} & V_N \\ \vdots & \vdots & \ddots & \vdots & \\ V_1 & V_2 & \dots & 0 & V_N \\ V_1 & V_2 & \dots & V_{N-1} & 0 \end{bmatrix}.$$

The players' costs per period of staying in the game are still $\vec{c} \equiv (c_1, \dots, c_N)$. The game with the interdependent valuations is denoted by $(N, N-1, U, \vec{c})$.

We still consider the equilibria with exponential strategies. Given strategy profile $\vec{\lambda} \equiv (\lambda_1, \dots, \lambda_N)$, the probability of ending the game by the exit of player l is $\lambda_l (\sum\limits_{n=1}^{N} \lambda_n)^{-1}$, and the expected duration of the game is $(\sum\limits_{n=1}^{N} \lambda_n)^{-1}$. Thus, the optimal strategy for player i is the solution to the following problem

$$\max_{\lambda_i \geq 0} \quad u_i(\lambda_i) = (\sum_{n=1}^{N} \lambda_n)^{-1} (\sum_{l \neq i} \lambda_l V_{il} - c_i).$$

It follows from the first-order conditions of the problem that if there exists an equilibrium with exponential strategies. Thus, the strategies satisfy

$$\sum_{l \neq i} \lambda_l V_{il} = c_i, \quad \text{i.e.,} \quad \vec{\lambda} \cdot U = \vec{c}.$$

The solution to the system of equations can be written as

$$\vec{\lambda}^* \equiv (\lambda_1^*, \ldots, \lambda_N^*) = (c_1, \ldots, c_N) \cdot U^{-1} = \vec{c} \cdot U^{-1}.$$

Proposition 4. *The war of attrition game with interdependent valuations $(N, N-1, U, \vec{c})$ has a nondegenerate mixed strategy equilibrium $\vec{\lambda}^* = \vec{c} \cdot U^{-1}$, conditional on $\vec{\lambda}^* > \vec{0}$. The expected payoffs to all players in the equilibrium are zero.*

Proof. Since the support of an exponential strategy is $[0, +\infty)$, we only need to show that given the other players' strategies $\vec{\lambda}_{-i}^*$, player i is indifferent to any pure strategy $t \in [0, +\infty)$. Player i faces a suppositional player with strategy $\sum_{n=-i} \lambda_n^*$. If player i exits at t, his expected payoff is

$$\begin{aligned}
&\int_0^t \Big(\sum_{l=-i} \frac{\lambda_l^*}{\sum_{n=-i} \lambda_n^*} V_{il} - xc_i\Big) dF_{\sum_{n=-i} \lambda_n^*}(x) - (1 - F_{\sum_{n=-i} \lambda_n^*}(t)) tc_i \\
&= \int_0^t \frac{c_i}{\sum_{n=-i} \lambda_n^*} dF_{\sum_{n=-i} \lambda_n^*}(x) - \int_0^t xc_i dF_{\sum_{n=-i} \lambda_n^*}(x) - (1 - F_{\sum_{n=-i} \lambda_n^*}(t)) tc_i \quad (\text{since } \vec{\lambda} \cdot U = \vec{c}) \\
&= F_{\sum_{n=-i} \lambda_n^*}(t) \frac{c_i}{\sum_{n=-i} \lambda_n^*} - [F_{\sum_{n=-i} \lambda_n^*}(t) tc_i - c_i \int_0^t F_{\sum_{n=-i} \lambda_n^*}(x) dx] - (1 - F_{\sum_{n=-i} \lambda_n^*}(t)) tc_i \\
&= (1 - e^{-t \sum_{n=-i} \lambda_n^*}) \frac{c_i}{\sum_{n=-i} \lambda_n^*} + c_i \Big(t + \frac{1}{\sum_{n=-i} \lambda_n^*} e^{-t \sum_{n=-i} \lambda_n^*} - \frac{1}{\sum_{n=-i} \lambda_n^*}\Big) - tc_i \\
&= 0.
\end{aligned}$$

Hence, $\vec{\lambda}^* = \vec{c} \cdot U^{-1}$ is an equilibrium mixed strategy profile of the game, and the equilibrium payoffs to all players are zero. ■

Proposition 4 suggests that interdependence in valuation does not fundamentally affect the nature of the game. With interdependence, a player's 'strength' cannot be represented by a simple expression. It depends on his valuations of the prize when different other players exit. The equilibrium strategies are less explicitly presented, which makes it harder to pronounce the strategies intuitively.

Finally, the game with interdependent valuations also has a set of partially degenerate equilibria. In such an equilibrium, one of the players, say $i \in \{1, \ldots, N\}$, exits immediately with probability $p \in (0,1)$ and chooses the exponential strategy λ_i^* (as described in Proposition 4) with probability $1-p$. The other players, $j \neq i$, choose the exponential strategy λ_j^* respectively.

3.2 Random defeat

There are often unpredictable events in 'battles' that may drive some players out and end the game early. Asako (2015) offers a short analysis of a two-player war of attrition with complete information and one-sided 'defeat'. The timing of the defeat follows an exponential distribution. I extend the idea to a war of attrition with N players. Moreover, I assume that the defeats may be associated with some additional payoffs.

In the base model $(N, N-1, \vec{V}, \vec{c})$, assume that player i may encounter a randomly arriving defeat at time x with exponential probability density $\eta_i e^{-\eta_i x}$, $\eta_i > 0$. A larger η_i implies that the defeat is more likely to occur in a given period. Write $\vec{\eta} \equiv (\eta_1, \ldots, \eta_N)$. The pure strategy equilibria of the game are virtually unaffected by the possible defeats, since the game ends immediately in those equilibria. Suppose that in the absence of defeat, the nondegenerate mixed strategy equilibrium of the game is $\vec{\lambda}^* = (\lambda_1^*, \ldots, \lambda_N^*)$. One can show that the game with defeat has an equilibrium with exponential strategies

$$\vec{\lambda}^d = \vec{\lambda}^* - \vec{\eta} = (\lambda_1^* - \eta_1, \ldots, \lambda_N^* - \eta_N)$$

as long as all the components are positive. A player with exponential strategy λ and defeat rate η can be viewed as a player with strategy $\lambda+\eta$ but no defeat risk. The proof is omitted here, as I will present a more general result later. Thus, when the players face the risk of defeat, they would subjectively behave tougher in a nondegenerate war of attrition. The expected duration of the game is unaffected since the exits of the players follow strategies $\vec{\lambda}^d+\vec{\eta}=\vec{\lambda}^*$.

A defeat in the real world often generates additional payoffs. For example, when an animal in combat loses due to injury, it suffers a loss in addition to the prize; in the exit game of a natural oligopoly, if a firm decides to drop out early because an unexpected outside option arises, then the 'defeat' yields a positive payoff. Let L_i denote player $i\in\{1,\ldots,N\}$'s additional 'loss' from a defeat. The 'loss' could be positive or negative. Write $\vec{L}\equiv(L_1,\ldots,L_N)$. Denote the game by $(N,N-1,\vec{V},\vec{c},\vec{\eta},\vec{L})$.

Given the players' mixed strategies $\lambda_1,\ldots,\lambda_N$ and the rates of defeat $\eta_1,\ldots,\eta_N$, player i loses the game if he is the first one to exit or encounter a defeat. Hence, player i's probability of winning the game is

$$1-(\lambda_n+\eta_n)(\sum_{n=1}^{N}(\lambda_n+\eta_n))^{-1}=[\sum_{n=-i}(\lambda_n+\eta_n)](\sum_{n=1}^{N}(\lambda_n+\eta_n))^{-1}.$$

His probability of losing the game by a defeat is $\eta_i(\sum_{n=1}^{N}(\lambda_n+\eta_n))^{-1}$, and the expected duration of the game is $(\sum_{n=1}^{N}(\lambda_n+\eta_n))^{-1}$. Hence, player i's strategy λ_i is the solution to problem

$$\max_{\lambda_i\geq 0}\quad u_i(\lambda_i)=\{[\sum_{n=-i}(\lambda_n+\eta_n)]V_i-\eta_iL_i-c_i\}(\sum_{n=1}^{N}(\lambda_n+\eta_n))^{-1}$$

The candidate mixed strategy equilibrium with exponential strategies $(\lambda_1,\ldots,\lambda_N)$ can be derived from the first-order conditions, which satisfies

$$\sum_{n=-i}(\lambda_n+\eta_n)=\frac{\eta_iL_i+c_i}{V_i},\quad i=1,\ldots,N.$$

If we view the 'cost per period' of player i as $\eta_iL_i+c_i$, and his 'displayed strategy' as $\lambda_i+\eta_i$, the structure of this system of equations is the virtually same as that without defeat.

Proposition 5. *The war of attrition game with random defeats* $(N, N-1, \vec{V}, \vec{c}, \vec{\eta}, \vec{L})$ *has a nondegenerate mixed strategy equilibrium* $\vec{\lambda}^* \equiv (\lambda_1^*, \ldots, \lambda_N^*)$ *in which*

$$\lambda_i^* = \frac{1}{N-1}(\sum_{n=1}^{N} \frac{\eta_n L_n + c_n}{V_n}) - \frac{\eta_i L_i + c_i}{V_i} - \eta_i, \quad i = 1, \ldots, N,$$

conditional on $\vec{\lambda}^* > \vec{0}$*. All players receive an expected payoff of zero in the equilibrium.*

Proof. From the first order conditions we immediately obtain the $\lambda_1^*, \ldots, \lambda_N^*$ described above. We only need to show that given the other players' strategies λ_{-i}^*, player i is indifferent to any pure strategy $t \in [0, +\infty)$. Under strategy profile $(\lambda_1^*, \ldots, \lambda_N^*)$, player i faces a suppositional 'player' with strategy $\sum\limits_{n=-i}(\lambda_n^* + \eta_n)$. To simplify the exposition, write

$$\gamma_i^* \equiv \sum_{n=-i}(\lambda_n^* + \eta_n) = \frac{\eta_i L_i + c_i}{V_i}, \quad i = 1, \ldots, N.$$

If player i exits at $t \in [0, +\infty)$, his expected payoff is

$$\begin{aligned}
&\int_0^t [\frac{\gamma_i^* V_i - \eta_i L_i}{\gamma_i^* + \eta_i} - x c_i] dF_{\gamma_i^* + \eta_i}(x) - (1 - F_{\gamma_i^* + \eta_i}(t)) t c_i \\
&= \int_0^t \frac{\gamma_i^* V_i - \eta_i L_i}{\gamma_i^* + \eta_i} dF_{\gamma_i^* + \eta_i}(x) - c_i \int_0^t x dF_{\gamma_i^* + \eta_i}(x) - (1 - F_{\gamma_i^* + \eta_i}(t)) t c_i \\
&= F_{\gamma_i^* + \eta_i}(t) \frac{\gamma_i^* V_i - \eta_i L_i}{\gamma_i^* + \eta_i} - c_i [t F_{\gamma_i^* + \eta_i}(t) - \int_0^t F_{\gamma_i^* + \eta_i}(x) dx] - (1 - F_{\gamma_i^* + \eta_i}(t)) t c_i \\
&= (1 - e^{-(\gamma_i^* + \eta_i)t}) \frac{\gamma_i^* V_i - \eta_i L_i}{\gamma_i^* + \eta_i} + c_i (t + \frac{e^{-(\gamma_i^* + \eta_i)t}}{\gamma_i^* + \eta_i} - \frac{1}{\gamma_i^* + \eta_i}) - c_i t \\
&= (1 - e^{-(\gamma_i^* + \eta_i)t}) \frac{\gamma_i^* V_i - (\eta_i L_i + c_i)}{\gamma_i^* + \eta_i} \\
&= 0.
\end{aligned}$$

Hence, $\vec{\lambda}^*$ is an equilibrium mixed strategy profile of the game, and the players' expected payoffs are zero in the equilibrium. ■

We may use $\frac{\eta_i L_i + c_i}{V_i}$ to measure the 'weakness' of player i. A player that is too weak may not be able to fight a nondegenerate war of attrition. When the equilibrium described in Proposition 5 exists, player i's probability of winning the game in the equilibrium is

$$[\sum_{n=-i}(\lambda_n^*+\eta_n)](\sum_{n=1}^{N}(\lambda_n^*+\eta_n))^{-1}=(N-1)(\sum_{n=1}^{N}\frac{\eta_n L_n+c_n}{V_n})^{-1}(\frac{\eta_i L_i+c_i}{V_i}).$$

Hence a 'weaker' player behaves tougher in the game and is more likely to win. The duration of the game follows an exponential distribution with a rate of

$$\sum_{n=1}^{N}(\lambda_n^*+\eta_n)=\frac{1}{N-1}(\sum_{n=1}^{N}\frac{\eta_n L_n+c_n}{V_n}).$$

Finally, let us take a quick look at the most general case in the current framework. In the war of attrition game in which N players compete for $N-1$ prizes, given the matrix of interdependent valuations U, the costs per period of staying in the game $c_1,\ldots,c_N$, the rates of random defeat $\eta_1,\ldots,\eta_N$, and the losses associated with the defeats $L_1,\ldots,L_N$, the Nash equilibrium with exponential strategies is

$$\vec{\lambda}^*\equiv(\lambda_1^*,\ldots,\lambda_N^*)=(\eta_1 L_1+c_1,\ldots,\eta_N L_N+c_N))U^{-1}-(\eta_1,\ldots,\eta_N),$$

conditional on $\vec{\lambda}^*>\vec{0}$. When there are N players competing for $N-k$ prizes, we may still have a nondegenerate equilibrium in which $k-1$ players exit immediately. The details are omitted here.

4 Conclusions

I consider a standard war of attrition game with complete information, in which N heterogeneous players compete for $N-K$ prizes. Each player can win at most one prize. A player is said to be 'stronger' if the ratio of his valuation of the prize to the cost of staying in the game is higher. In the base model in which $K=1$, I explicitly characterize a subgame perfect mixed strategy equilibrium of the game, in which all players adopt strategies with exponential distributions. All players receive expected payoffs of zero in the equilibrium. The equilibrium may not exist when there are three or more players and the weakest player is too weak.

However, as long as the equilibrium exists, the weaker players behave tougher and are more likely to win. The equilibrium, if exists, is the unique equilibrium in which the strategies follow atomless distributions and have supports of the entire time horizon. The game may also have partially degenerate equilibria in which one of the players exits immediately with positive probability. In the case where $K \geq 2$, the game may have nondegenerate equilibria in which $K-1$ players exit immediately. The findings can be applied to an 'all-pay Japanese auction' with complete information.

I also considered two extensions of the base model. The first is that players have interdependent valuations of the prizes, i.e., a player's valuation of a prize depends on which other player exits. Although there is no easy way to measure a player's 'strength' in this case, a memoryless mixed strategy equilibrium can still be characterized. The second is that the players face randomly arriving 'defeats'. It was found that the players tend to choose tougher strategies, since random defeat serves as part of their exit decisions. The losses associated with the defeats also tend to make the players tougher in a nondegenerate war of attrition.

Of the equilibria mentioned in this paper, namely the pure strategy, the partially degenerate, and the nondegenerate equilibria, the nondegenerate equilibria lead to the 'fairest' payoffs. Unfortunately, it is also the least efficient one. There appears to be a conflict between fairness and efficiency in the game. I conjecture that the relatively efficient pure strategy equilibria may be infeasible in a society that places a high value on fairness. This may be a good topic for future research, as it addresses an important issue in a society where decision-making is decentralised.

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