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Tuesday 1st December, 2015

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1 Introduction

The idea of using derivatives of market volatility to manage financial risk can be traced back to far before the Chicago Board of Option Exchange (CBOE) developed its Volatility Index (VIX). Brenner and Galai (1989) introduced a volatility index (Sigma index) and discussed derivatives such as options and futures on this index. Following this idea, Fleming et al. (1995) introduced the old version of the VIX, which depends on the inversion of the Black-Scholes formula. However, standardized derivative contracts on VIX were not available until the CBOE calculated VIX on a model-free basis in 2003. Since the introduction of VIX futures/options in 2004, volatility derivatives have become a set of popular derivatives in the market especially after the subprime crisis.

To answer the demand for pricing the volatility derivatives in this fast-growing market, several models were proposed, such as the binomial process for volatility option pricing (Brenner and Galai (1997)), the square root mean-reverting process for volatility futures and options (Grunbichler and Longstaff (1996)), the Heston[1] model-based process for volatility futures (Zhang and Zhu (2006)), etc. Zhu and Zhang (2007) also proposed a non-arbitrage model for VIX futures based on VIX term structures. In consideration of possible jumps in log-returns, Duffie et al. (2000) proposed an affine jump-diffusion process for log-returns that soon became a new benchmark process in the asset pricing literature. On the basis of this process and its modifications, Lin (2007) proposed a stochastic volatility model with simultaneous jumps in both returns and volatility to price VIX futures and yielded an approximation

[1]Proposed by Heston (1993) for option pricing.

formula. Sepp (2008) added jumps into the square root mean-reverting process to price VIX options. Zhang et al. (2010) included additional stochastic long-run variance into the jump square root mean-reverting process and linked VIX and VIX futures quotes. By adding jumps to the Heston model, Zhu and Lian (2012) found an analytical pricing formula for VIX futures.

Despite the development of literature on the stochastic process for volatility derivatives, little attention has been paid to discrete-time models. The literature mainly focuses on equity option pricing, such as Duan (1995), Duan (1999), Heston and Nandi (2000), Duan et al. (2005), Christoffersen et al. (2008), Christoffersen et al. (2014), etc.[2] To our best knowledge, little (if any) literature exists on the pricing of VIX derivatives under the GARCH framework. One possible reason is that the conventional local risk-neutral valuation relationship (LRNVR) only responds to equity risk premium, and there is no room for an independent variance risk premium within a single shock in the GARCH models. To overcome this problem, the recent literature estimates parameters with information from both the underlings and the risk-neutral measures such as option prices and the VIX. For example, Hao and Zhang (2013) suggested that joint estimates can greatly improve the GARCH model's ability of fitting the VIX. Kanniainen et al. (2014) show that joint estimation with VIX data can greatly improve the GARCH model's option pricing ability. These results indicate that the GARCH model can be used in derivative pricing topics with an appropriate estimation method.

To fill this potential gap in the literature on volatility derivatives pricing, we focus on the issue of pricing VIX futures with discrete-time GARCH-type models. One appealing advantage of GARCH-type models is the convenience of parameter estimation. Unlike stochastic volatility models with unobservable shocks, the shocks in the GARCH model are observable, and the estimation is straightforward for implementation with maximum likelihood estimation. For large futures samples with substantial cross-sectional dimension over a long period, it is important to use a less computationally demanding model to implement the estimation discussed above jointly using both the VIX and VIX futures prices. GARCH models, in our opinion, provide a good foundation from this perspective.

In this paper, we discuss VIX futures pricing under the classical discrete-time Heston-Nandi GARCH model. An explicit pricing formula is provided via an integration of a transformed moment-generation function of conditional volatility. Several estimation methods, in terms of the data used, are provided, and their performance is investigated with the use of market data. Among them, the estimation by jointly using the VIX and VIX futures prices yields satisfying pricing performance and good fit for both the implied VIX and VIX future prices.

The remainder of the paper is structured as follows. Section 2 discusses the model's pricing formula for implied VIX and VIX future prices. Section 3 provides a series of estimation methods. Section 4 summarizes the main empirical results, and Section 5 presents our conclusions.

[2] Christoffersen et al. (2012) provides an extensive review of equity option pricing with GARCH models.

2 The Model

2.1 Heston-Nandi GARCH model and risk neutralization

We assume the return of the S&P 500 index follows the Heston-Nandi GARCH model under the physical measure $\mathbb{P}$:

$$\begin{aligned} r_{t+1} &= r + \lambda h_{t+1} - \frac{1}{2}h_{t+1} + \sqrt{h_{t+1}}\varepsilon_{t+1} \\ h_{t+1} &= \omega + \beta h_t + \alpha(\varepsilon_t - \delta\sqrt{h_t})^2 \end{aligned} \tag{2.1}$$

where ε_t follows a standard normal distribution, r is the risk-free interest rate, r_t is the log-return of the asset, and λ is the equity premium parameter associated with the conditional variance, which allows the average spot return to depend on the level of risk. The variance equation nests the leverage effect and volatility clustering effect, which are common in the financial market.

Under the LRNVR proposed by Duan (1995), we have the following risk-neutral ($\mathbb{Q}$) dynamics:

$$\begin{aligned} r_{t+1} &= r - \frac{1}{2}h_{t+1} + \sqrt{h_{t+1}}\varepsilon^*_{t+1} \\ h_{t+1} &= \omega + \beta h_t + \alpha(\varepsilon^*_t - \delta^*\sqrt{h_t})^2 \end{aligned} \tag{2.2}$$

where $\varepsilon^*_t = \varepsilon + \lambda\sqrt{h_t}$, $\delta^* = \delta + \lambda$. The unconditional expectation of h_t under the $\mathbb{Q}$ measure is

$$\sigma_h^2 = E^{\mathbb{Q}}[h_t] = \frac{\widetilde{\omega}}{1-\widetilde{\beta}} \tag{2.3}$$

where $\widetilde{\omega} = \omega + \alpha$ and $\widetilde{\beta} = \beta + \alpha(\delta+\lambda)^2$.

2.2 The model-implied VIX formula

According to the CBOE and related papers such as Hao and Zhang (2013), the VIX can be calculated as the annualized arithmetic average of the expected daily variance over the following month, which is

$$\left(\frac{VIX_t}{100}\right)^2 = \frac{1}{n}\sum_{k=1}^{n} E_t^{\mathbb{Q}}[h_{t+k}] \times AF \tag{2.4}$$

where h_t is the instantaneous daily variance of the return of the S&P 500. AF is the annualizing factor that converts daily variance into annualized variance by holding the daily variance constant over a year. For simplicity, we define

$$V_t = \frac{1}{22}\sum_{k=1}^{22} E_t^{\mathbb{Q}}[h_{t+k}] \tag{2.5}$$

and thus, $V_t = \frac{1}{252}(\frac{VIX_t}{100})^2$ is a proxy for VIX_t in terms of daily variances[3].

The affine structure of the Heston-Nandi GARCH model provides the following linear relationship between the model-implied VIX and conditional variance.

PROPOSITION 1. *If the S&P 500 return follows the Heston-Nandi GARCH model presented in (2.2), then the implied volatility at time t is a linear function of the conditional volatility of next period*

$$V_t = \Psi + \Gamma h_{t+1} \tag{2.6}$$

where $\Gamma = \frac{1-\widetilde{\beta}^n}{n(1-\widetilde{\beta})}$ *and* $\Psi = \frac{\widetilde{\omega}}{1-\widetilde{\beta}}(1-\Gamma)$.

Proof of this can be found in Hao and Zhang (2013). The model-implied VIX_t is then given by annualizing the conditional daily standard deviation:

$$VIX_t = 100 \times \sqrt{252 V_t} = 100 \times \sqrt{252(\Psi + \Gamma h_{t+1})} \tag{2.7}$$

2.3 The explicit formula of VIX futures price

The advantage of using the Heston-Nandi GARCH model is that it ensures a closed-form solution to the VIX futures price. This enables estimation methods that aim to minimize the VIX futures pricing error. For the purpose of simplification, we rewrite (2.7) as:

$$VIX_t = 100 \times \sqrt{252 V_t} = 100 \times \sqrt{a + b h_{t+1}}$$

where a and b are defined as $a = 252 \times \Psi$ and $b = 252 \times \Gamma$.

The price of the VIX futures can be interpreted as its conditional expectation at maturity time T evaluated at the current time t under risk-neutral measures, which allows the following expression given by Zhu and Lian (2012):

$$F(t,T) = E_t^{\mathbb{Q}}[VIX_T] = E_t^{\mathbb{Q}}[100\sqrt{a + b h_{T+1}}] = \frac{100}{2\sqrt{\pi}} \int_0^{\infty} \frac{1 - e^{-sa} E_t^{\mathbb{Q}}[e^{-sbh_{T+1}}]}{s^{3/2}} ds \tag{2.8}$$

where t is the current date and $T - t$ is the time to maturity.

The last term $E_t^{\mathbb{Q}}[e^{-sbh_{T+1}}]$ can be expressed as the moment-generating function of the conditional variance at time $T+1$. Under the Heston-Nandi GARCH model, this allows a closed-form solution as stated in the following proposition.

[3]This means that we assume 22 days a month and 252 days a year, which is quite common in terms of the numbers of trading days.

PROPOSITION 2. *The moment-generating function of the conditional variance h_{t+m} (m days from now) at time t is exponentially affine in h_{t+1}, and it allows the following expression*

$$f(\phi, m, h_{t+1}) = E_t^{\mathbb{Q}}[e^{\phi h_{t+m+1}}] = e^{C(\phi,m)+H(\phi,m)h_{t+1}} \tag{2.9}$$

where the functions $C(\phi, m)$ and $H(\phi, m)$ are given by an iterative relationship

$$\begin{aligned} C(\phi, n+1) &= C(\phi, n) - \frac{1}{2}\ln(1 - 2\alpha H(\phi, n)) + \omega H(\phi, n) && (2.10) \\ H(\phi, n+1) &= \frac{\alpha\delta^* H(\phi, n)}{1 - 2\alpha H(\phi, n)} + \beta H(\phi, n) && (2.11) \end{aligned}$$

with initial condition

$$C(\phi, 0) = 0, \quad H(\phi, 0) = \phi \tag{2.12}$$

The parameters are defined in the Heston-Nandi GARCH model.

Proof: See the Appendix.

Hence, the VIX futures price under the Heston-Nandi GARCH model can be calculated by:

$$F(t, T) = \frac{100}{2\sqrt{\pi}} \int_0^\infty \frac{1 - e^{-sa} f(-sb, T-t, \frac{(VIX_t/100)^2 - a}{b})}{s^{3/2}} ds \tag{2.13}$$

The function $f(\cdot)$ is defined in proposition 2.

3 Data and Estimation

Basically, we have three different time series: the log-return series of the S&P 500, the quote series of the CBOE VIX, and the series of VIX futures prices for different maturities across time. The first two series are collected from Yahoo Finance, and the last series is collected from the CBOE's website. All series range from 2004.3.26 to 2013.12.18 and contain 2451 returns/VIX and 14501 VIX futures prices. A summary of the dataset is shown in Table I and II.

[Insert Table I here]

[Insert Table II here]

These three series form an information cascade with the log-returns at the bottom level and the VIX futures at the top level. Following this cascade, three "single series" estimation methods can be applied.

The first approach, which is the most straightforward, is to run a maximum likelihood estimation with the bottom-level data (S&P 500 returns) only. The likelihood function is:

$$\ln L_R = -\frac{T}{2}\ln(2\pi) - \frac{1}{2}\sum_{t=1}^{T}\Big(\ln(h_t) + [R_t - r - \lambda h_t + \frac{1}{2}h_t]^2/h_t\Big) \tag{3.1}$$

where h_t is updated by the conditional variance process. Such estimation provides us with parameters under physical dynamics. Due to the simplicity of both the Heston-Nandi GARCH model and the LRNVR, this estimation can obtain all parameters needed to recover its risk-neutral dynamics and, therefore, is sufficient to calculate the model-implied VIX and VIX futures prices.

The second approach is to calibrate[4] with the mezzanine level data (VIX) only. Assume the following distribution

$$VIX^{Mkt} = VIX^{Imp} + u, \quad u \sim i.i.d.N(0, s_V^2)$$

The s_V^2 is estimated using the sample variance of pricing errors. Then we have the log-likelihood function:

$$\ln L_V = -\frac{T}{2}\ln(2\pi \hat{s_V}^2) - \frac{1}{2\hat{s_V}^2}\sum_{t=1}^{T}(VIX^{Mkt} - VIX^{Imp})^2 \tag{3.2}$$

This method focuses on the fitness of the VIX only instead of the S&P return. Because the VIX is the most direct underlying asset of the VIX futures, we suppose that it contains more information than the S&P 500 returns. For example, as shown in a previous study (Hao and Zhang (2013), etc.), the variance risk premium contained in the VIX cannot be obtained by fitting the model using the return data of the S&P 500 only. Frijns et al. (2015) documented a strong Granger causality from VIX futures to VIX than the other way around. Therfore, it might be insufficient to consider VIX data only.

The third approach is to calibrate with the top-level data (VIX futures prices) only. This approach is direct and common in practice regardless of rationality: in terms of pricing error, this approach is supposed to have the best pricing performance because its objective function only concerns the prices of VIX futures. However, this approach may lead to great distortion when we use the calibrated parameters for implied VIX calculation. The corresponding likelihood function is:

$$\ln L_F = -\frac{T}{2}\ln(2\pi \hat{s_F}^2) - \frac{1}{2\hat{s_F}^2}\sum_{t=1}^{T}(Fut^{Mkt} - Fut^{Mod})^2 \tag{3.3}$$

Besides the above three straightforward estimation approaches, we also apply additional methods with mixed information. The fourth approach is to run a joint estimation with the lower two levels of data (S&P 500 returns and the VIX). This kind of joint method is popular in the current pricing

[4] Although we are using a maximum likelihood estimation notation, this method is essentially a calibration method.

literature that aims to better reconcile $\mathbb{P}$ and $\mathbb{Q}$ measure information[5]. The parameters can be obtained by maximizing the joint log-likelihood function:

$$\ln L_{VR} = \ln L_V + \ln L_R \tag{3.4}$$

The last estimation approach is to run a joint estimation with the upper two levels of data (the VIX and VIX futures prices). That is, we estimate parameters by maximizing the joint log-likelihood function as:

$$\ln L_{VF} = \ln L_V + \ln L_F \tag{3.5}$$

Compared with the estimation methods provided above, this method, by taking into account the fitting errors of both the VIX and VIX futures prices together, seeks a good balance between the underling asset (VIX) and the derivatives (VIX futures).

To insure the stationarity of the volatility process, the following constraint on parameters is imposed in addition to the positivity constraint of all parameters.

$$\beta + \alpha\delta^{*2} < 1$$

Note that $\delta^* = \delta + \lambda$. The positivity of λ and δ implies that this $\mathbb{Q}$ measure stationary condition also means $\mathbb{P}$ measure stationarity.

4 Empirical Results

4.1 Parameters

In Table III, we present our parameter estimation results for the Heston-Nandi GARCH model using the different estimation approaches. The first row denotes the estimation approach we use. For example, the "Return" column denotes the estimation by S&P 500 returns data only. The parameter π here is defined as $\beta + \alpha\delta^*$, which measures the persistence of the filtered conditional volatility. The last five rows present the values of the log-likelihood functions by each estimation approach in which the bolded value is the conventional reported likelihood value[6]. Robust standard errors are provided in parentheses.

[Insert Table III here]

[5] See Kanniainen et al. (2014) etc.

[6] The bolded diagonal elements denote the ones to be maximized, whereas the off-diagonal elements denote the likelihood value of corresponding parameters in the column evaluated with the likelihood function in the row. For example, the L_R value for VIX is the value of L_R evaluated by parameters calibrated by VIX. It is clear that each set of parameters maximizes the corresponding likelihood function.

The most notable finding in Table 3 is the obvious parameter distortion in the estimation by VIX futures prices. The β increases significantly and becomes very close to 1, which is its upper limit. Contrastingly, the δ^* drops largely from around 300 to 5.65. These parameters are relatively robust to initial values because the parameters from the last column (which also involves fitting VIX futures) do provide a significantly larger L_F. Such parameter distortion indicates a great smoothness in the model-implied VIX. With such large β values, the conditional volatility at time t nearly totally determines the conditional volatility at time $t+1$, whereas the "shock" at time t has little effect (much weaker leverage effect δ^*). Other notable findings among "single series" estimations are: 1) a significant rise of δ^* from the return-based estimate to the VIX-based estimate; 2) a significant rise of persistent parameter π from $\mathbb{P}$ information-based parameters (return only) to $\mathbb{Q}$ information-based parameters (VIX index and VIX futures). These two findings are consistent with the existing literature (e.g. Christoffersen et al. (2014), Kanniainen et al. (2014) etc.).

For mixed information estimations, parameters by joint estimation using S&P 500 returns and VIX are very close to those estimated by VIX only. Thus, incorporating the return data with VIX data does not provide much additional information for parameter estimation. However, this is not the case for joint estimation using the VIX index and VIX futures whereby joint estimates significantly differ from those calibrated only from VIX futures or the VIX index itself. The following subsections are provided to investigate the model's ability to replicate the VIX index and price VIX futures in which a desirable parameter set should perform well in both aspects[7].

4.2 Model-implied VIX

Table IV summarizes the performance of different estimation methods for the model-implied VIX. The first four columns are related to errors, which are defined as the market VIX minus the corresponding model-implied VIX. Conventional measures are provided, namely, the mean error (ME), the root mean square error (RMSE), the mean absolute error (MAE) and the standard deviation of error (StdErr). The last column is the correlation coefficient (Corr) between the model-implied VIX and the market VIX. Plots on both series are also provided[8] in Figure 1.

[Insert Table IV here]

[Insert Figure 1 here]

Not surprisingly, the model-implied VIX generated by parameters based on S&P 500 returns only significantly underestimates the VIX index. This reflects the fact that the single-shock GARCH-type

[7] A more desirable case is that a model, with one set of parameters, can replicate all three levels of data: S&P 500 returns, VIX and VIX futures. Unfortunately, the existing literature indicates that it is hard for GARCH-type models to replicate both $\mathbb{P}$ and $\mathbb{Q}$ dynamics. Therefore, this paper only focuses on the VIX and VIX futures and leaves the additional fit of returns for further research.

[8] The graph for VIX and return joint estimation is almost identical to that for VIX only and is available upon request.

model under LRNVR can only capture equity risk premium and leaves the volatility risk premium unattended. When parameters are calibrated by the VIX index or another information source under risk-neutral dynamics (where both premiums are already taken into account in the dataset), the underestimation is no longer profound.

The model-implied VIX based only on the estimation by VIX futures prices has an even higher error than that based only on returns. This fact is more obvious in Figure 1(c) in which the model-implied VIX is just a long run smoothed of the actual data. This confirms the results that we have discussed in model estimation: parameters lead to great distortion of underlying dynamics.

For joint estimation using VIX and VIX futures, although the parameters here are obviously different from those estimated by VIX only, the fitting performance for the implied VIX is similar, with a RMSE of 4.94 (combination of VIX and VIX futures) compared with 4.60 (VIX only). Moreover, compared with the parameters using VIX futures data only, the joint parameters show a significant improvement regarding implied VIX fitting, with the RMSE increasing from 6.9 to 4.9. This indicates that the parameters by joint estimation of VIX and VIX futures are more desirable.

4.3 VIX Futures Prices

Table V summarizes the performance of the different methods for VIX futures pricing. The meanings of each column are as defined in the previous subsection. Not surprisingly, the parameters based on VIX futures only provide the best pricing performance. Among the other approaches, parameters estimated by return only show the worst RMSE performance. The table also shows a severe underestimation for VIX futures prices. Parameters by VIX only and the combination of return/VIX demonstrate better pricing performance for VIX futures compared with that by return only. Again, joint estimation of VIX and VIX futures shows a significant improvement over that of VIX-only estimation, and the results are very close to the prices estimated by the VIX futures-only approach.

[Insert Table V here]

For more detailed information on RMSE, we present Table VI in which the RMSE of pricing errors is summarized by different VIX levels, basis[9] levels and time to maturity.

[Insert Table VI here]

In addition to the above findings, the model performs better when the maturity is shorter and the basis is smaller. The RMSE increases when the VIX level is greater than 15. These findings are intuitive because these cases include either futures, which are similar to the underlings, or the underlings themselves, which are less volatile. However, the RMSE is also significantly higher when the VIX is low. The reason for this lies in the fact that the parameters are trained to fit VIX series for full samples,

[9]Basis = VIX index - VIX futures price.

whereas peaks of the VIX around a subprime crisis tend to be overlooked under the square root loss function.

Following Zhu and Lian (2012), we also provide the term structure[10] of the model-implied VIX futures along with its market counterpart in Figure 2. The blue lines indicate the market's term structure, the red lines indicate the term structure estimated jointly by VIX/VIX futures, and the green lines correspond to the estimation by VIX only. Instead of evaluating the term structure on the mean of VIX in the full sample, we evaluate the structure with a partition of futures data with respect to their underlying VIX levels. The main concern of this modification is to reconcile the fact that our sample spans a period during which the VIX changed violently. In particular, we divided the futures into two groups: the first group contains those with an underlying VIX level of less than 25, which is denoted as the "Normal VIX level" group. The second group contains the futures with an underlying VIX level greater than 25, denoted as the "High VIX level" group[11].

[Insert Figure 2 here]

Before focusing on the comparison, it is important to notice that the structure is totally different when the underlying VIX is high as it shows a downward slope instead of the commonly reported full sample upward-sloped curve. This is a natural result when volatility follows a mean reversion process: the high VIX level is much higher than the mean VIX level; therefore, we expect the future VIX level to fall instead of rise. The term structure implied by the VIX-only parameters has a significant larger error of overestimation when volatility is normal and underestimation when volatility is high. A possible explanation for this lies in the fact that the VIX dynamics implied by futures prices are much more persistent than the VIX dynamics themselves[12]. The higher persistence of volatility means that we expect a lower reversion speed for volatility and a flatter term structure for VIX futures.

These findings shed light on the role of the parameters and the reason why joint estimates work better. First, persistence parameter π is important for VIX futures pricing. A better pricing result requires that π be very close to one. However, the leverage parameter δ and the parameter of lagged volatility β are important for fitting VIX dynamics. A better fit for implied VIX usually requires a lower β and a higher δ to apply more weight to current shocks rather than historical volatility. When joint estimation was applied, the VIX futures data trained the model to have a high level of π, whereas the VIX data guided the model to achieve this goal by increasing β while maintaining a high level of δ. Without the help of the VIX data, as with the VIX futures-only case, the model will easily get trapped with an extremely high β and low δ.

[10] We divide the future data into groups according to their maturities. Each group is 10 days long. The term structure is defined as the average of market/model prices within each group and those plotted against maturities. The graph for the joint estimation of VIX and return is almost identical to that for VIX only and is available upon request.

[11] The first group contains some 77% of all constructs. VIX = 25 is also a cut-off point when we discuss RMSE for pricing errors.

[12] Recall the results from parameter estimation when the futures-only method yields a high β and low δ.

5 Conclusion

In this paper, we discuss VIX futures pricing under the classical discrete-time Heston-Nandi GARCH model. Under the LRNVR, an explicit pricing formula can be found via an integration of the transformed moment generation function of conditional volatility. In terms of the information cascade of data, several estimation approaches are provided, and their performance is investigated with the use of market data. In line with the existing literature, the bottom-level return data provide little information on VIX futures, whereas the mezzanine-level VIX index data provide much more. Although the calibration with respect to the top-level VIX futures data provides minimum pricing errors, it leads to great distortion in the model-implied VIX. However, the joint information from the VIX index and VIX futures can provide similar pricing performance on VIX futures without distortion on the dynamic of the model-implied VIX. A brief discussion on model parameters is provided to address the importance of the balance of β and δ in achieving a π with higher persistence. The results confirm that the Heston-Nandi GARCH model with appropriate information sources under the $\mathbb{Q}$ measure can fit both VIX future prices and VIX index dynamics. However, simply adding returns information cannot lead to a good fit of all three levels. Such finding indicates the importance of searching for a new model with the ability to fit both $\mathbb{P}$ and $\mathbb{Q}$ dynamics.

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Appendix

Proof of Proposition 2

Proof. Suppose the moment-generating function has the following form:

$$E_t^{\mathbb{Q}}[e^{\phi h_{t+m+1}}] = e^{C(\phi,m)+H(\phi,m)h_{t+1}}$$

Then by the initial condition, we have

$$C(\phi, 0) = 0, \quad H(\phi, 0) = \phi$$

We also suppose the following recursion relationship:

$$\begin{aligned} C(\phi, m+1) &= C(\phi, m) + C_J(H(\phi, m)) \\ H(\phi, m+1) &= H_J(H(\phi, m)) \end{aligned}$$

We will see under the Heston-Nandi GARCH model that we can have a closed-form solution for $C_J(\cdot)$ and $H_J(\cdot)$.

We have

$$E_t^{\mathbb{Q}}[e^{\phi h_{t+m+2}}] = e^{C(\phi,m+1)+H(\phi,m+1)h_{t+1}}$$

We also have

$$\begin{aligned} E_t^{\mathbb{Q}}[e^{\phi h_{t+m+2}}] &= E_t^{\mathbb{Q}}[E_{t+1}^{\mathbb{Q}}[e^{\phi h_{t+m+2}}]] \\ &= E_t^{\mathbb{Q}}[e^{C(\phi,m)+H(\phi,m)h_{t+2}}] \end{aligned}$$

Let

$$s = H(\phi, m)$$

Then we solve

$$\begin{aligned}
E_t^{\mathbb{Q}}[e^{sh_{t+2}}] &= e^{s\omega+s\beta h_{t+1}+s\delta^{*2}\alpha h_{t+1}} E_t^{\mathbb{Q}}[e^{s\alpha\varepsilon_{t+1}^{*2}-2s\alpha\delta^*\sqrt{h_{t+1}}\varepsilon_{t+1}^*}] \\
&= e^{s\omega+s\beta h_{t+1}+s\delta^{*2}\alpha h_{t+1}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} e^{s\alpha x^2-2s\alpha\delta^*\sqrt{h_{t+1}}x} dx \\
&= e^{s\omega+s\beta h_{t+1}+s\delta^{*2}\alpha h_{t+1}} \Big(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\frac{1}{\sqrt{1-2s\alpha}}} e^{-\frac{(x+\frac{2s\alpha\delta^*\sqrt{h_{t+1}}}{1-2s\alpha})^2}{2(\frac{1}{1-2s\alpha})}} dx \Big) \frac{1}{\sqrt{1-2s\alpha}} e^{\frac{2s^2\alpha^2\delta^{*2}h_{t+1}}{1-2s\alpha}}
\end{aligned}$$

The integral part is 1, so we have

$$E_t^{\mathbb{Q}}[e^{sh_{t+2}}] = e^{s\omega-\frac{1}{2}\ln(1-2s\alpha)} e^{\left(s\beta+\frac{s\alpha\delta^{*2}}{1-2s\alpha}\right)h_{t+1}}$$

By comparison, we have

$$\begin{aligned}
C_J(s) &= s\omega - \frac{1}{2}\ln(1-2s\alpha) \\
H_J(s) &= s\beta + \frac{s\alpha\delta^{*2}}{1-2s\alpha}
\end{aligned}$$

Thus, we have obtained the closed-form recursion relationship for the moment-generating function. □

TABLE I

Summary statistics for S&P 500 returns and VIX

	Number	*Mean*	*Std*	*Max*	*Min*	*Skew*	*Kurt*
Return	2451	0.0002	0.01	0.11	-0.09	-0.33	13.90
VIX	2451	20.27	10.00	80.86	9.89	2.29	9.72

TABLE II

Summary statistics for VIX futures

	Number	*Mean*	*Std*	*Max*	*Min*	*Skew*	*Kurt*
VIX							
< 15	3809	15.54	2.51	27.60	10.37	1.44	6.42
$15 \sim 20$	4496	21.46	3.37	31.00	13.50	0.05	2.46
$20 \sim 25$	2848	25.52	2.98	33.35	17.38	0.30	2.42
$25 \sim 30$	1339	27.96	2.92	34.55	21.10	-0.13	2.17
> 30	2009	36.02	6.98	66.23	22.15	0.92	4.13
Basis							
< -6	692	36.09	8.40	66.23	22.15	0.59	3.10
$-6 \sim -3$	520	31.29	9.23	63.88	14.42	0.53	3.33
$-3 \sim +3$	7311	21.43	7.56	63.71	10.37	1.16	4.49
$3 \sim 6$	3899	21.99	4.89	51.13	13.08	0.46	3.44
> 6	2079	26.20	3.26	34.20	17.35	0.13	2.62
Maturity							
< 50	3641	21.42	8.87	66.23	10.37	1.70	6.53
$50 \sim 100$	3365	23.21	7.77	59.77	12.21	1.15	4.74
$100 \sim 150$	3082	24.06	6.97	50.58	13.16	0.70	3.40
$150 \sim 200$	2785	24.77	6.36	45.26	13.52	0.41	3.03
> 200	1628	23.90	5.50	38.80	14.32	0.17	2.44

TABLE III

Parameters of the Heston-Nandi GARCH model under different methods

	Return	*VIX*	*Return + VIX*	*VIX future*	*VIX + VIX future*
β	0.7638	0.7064	0.6963	0.9954	0.7939
	(0.026)	(0.001)	(0.002)	(0.000)	(0.003)
α	3.4108E-6	2.3415E-6	2.4053E-6	1.2139E-6	1.4124E-6
	(9.933E-8)	(3.527E-9)	(1.238E-8)	(1.100E-9)	(2.136E-8)
$\delta^{\star}$	249.3500	349.0718	350.1333	5.6549	377.5120
	(17.094)	(0.987)	(1.957)	(0.007)	(5.838)
h_0	1.2006E-4	2.8403E-4	2.7323E-4	2.6789E-4	2.9545E-4
	(1.114E-4)	(8.634E-6)	(3.994E-6)	(2.785E-6)	(3.301E-6)
π	0.9759	0.9918	0.9912	0.9955	0.9952
Log-likelihood					
L_R	**7895**	7810	7820	7480	7740
L_V	-8152	**-7218**	-7222	-8227	-7394
L_{VR}	-258	593	**598**	-747	346
L_F	-51214	-40656	-40932	**-38977**	-39151
L_{VF}	-59366	-47874	-48154	-47204	**-46545**

Robust standard errors are in parentheses.

TABLE IV

Summary of model-implied VIX

	ME	*RMSE*	*MAE*	*StdErr*	*Corr*
Return	3.7506	6.7337	4.4262	5.5936	0.8929
VIX	-0.1270	4.5990	3.3600	4.5982	0.8965
VIX futures	-1.6971	6.9423	5.2076	6.7331	0.8179
Return + VIX	0.0524	4.6076	3.3671	4.6082	0.8967
VIX + VIX futures	-0.2551	4.9424	3.3236	4.9368	0.8818

TABLE V

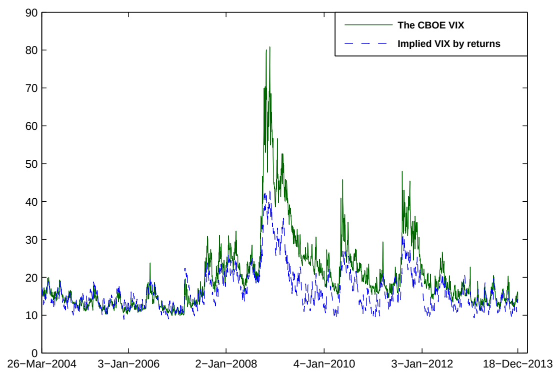
Summary of VIX futures pricing

	ME	*RMSE*	*MAE*	*StdErr*	*Corr*
Return	5.8400	8.2716	6.3069	5.8579	0.6448
VIX	-0.0471	3.9938	3.1188	3.9937	0.8540
VIX futures	0.0760	3.5571	2.6242	3.5564	0.8832
Return + VIX	0.1843	4.0705	3.1860	4.0664	0.8500
VIX + VIX futures	0.1446	3.5999	2.6925	3.5971	0.8793

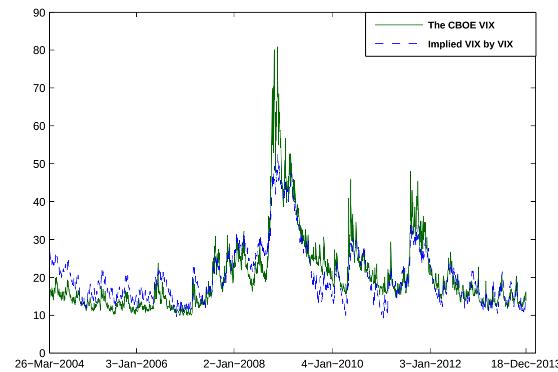
TABLE VI

RMSE for VIX futures pricing by VIX, basis and maturity

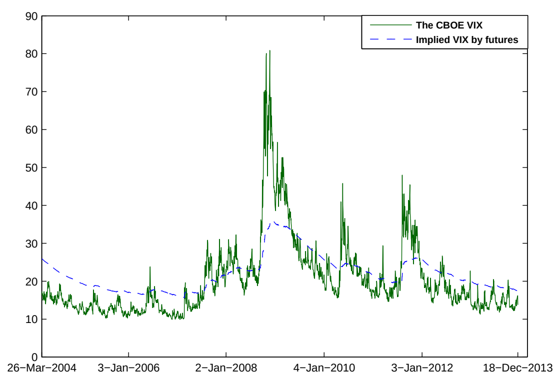
VIX	< 15	$15 \sim 20$	$20 \sim 25$	$25 \sim 30$	> 30
Return	2.0858	5.9416	8.7581	10.1878	15.1221
VIX	4.6975	2.5812	3.3476	3.8200	5.7246
VIX futures	3.2469	2.6568	3.2959	3.2811	5.7421
Return + VIX	4.5915	2.5911	3.5111	4.0465	6.0174
VIX + VIX futures	3.4027	2.6586	3.3805	3.4116	5.6404
Basis	< -6	$-6 \sim -3$	$-3 \sim +3$	$3 \sim 6$	> 6
Return	15.8516	13.0987	6.7470	6.8557	10.1222
VIX	6.4425	4.5313	3.6728	3.9013	4.0442
VIX futures	8.8647	3.0011	2.5538	3.0840	4.3779
Return + VIX	6.4705	4.8595	3.7282	3.9160	4.2557
VIX + VIX futures	8.4467	3.0431	2.6817	3.2424	4.4045
Maturity	< 50	$50 \sim 100$	$100 \sim 150$	$150 \sim 200$	> 200
Return	4.2043	7.9802	9.5198	10.2119	9.1597
VIX	2.4064	3.8507	4.4172	4.7273	4.7412
VIX futures	2.3049	3.5500	3.9429	3.9968	4.1703
Return + VIX	2.4254	3.9127	4.5099	4.8644	4.7880
VIX + VIX futures	2.3157	3.5798	3.9804	4.0704	4.2400



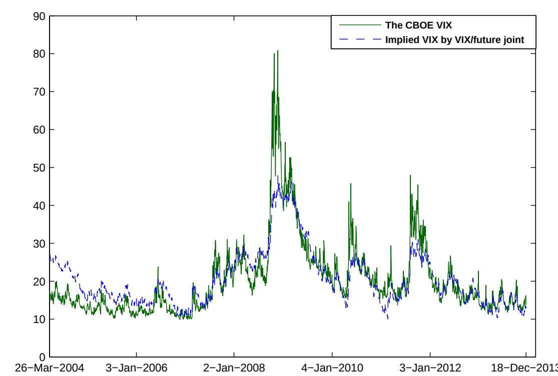
(a) Return only



(b) VIX only



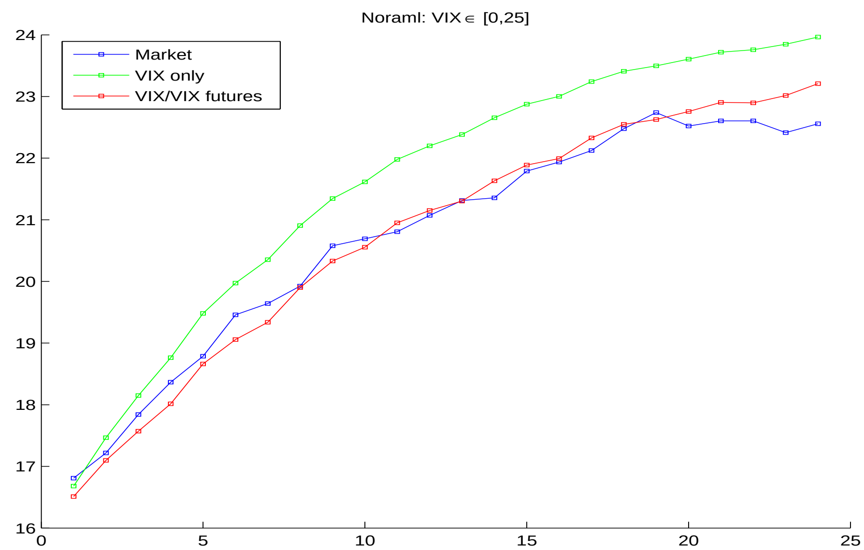
(c) VIX futures only



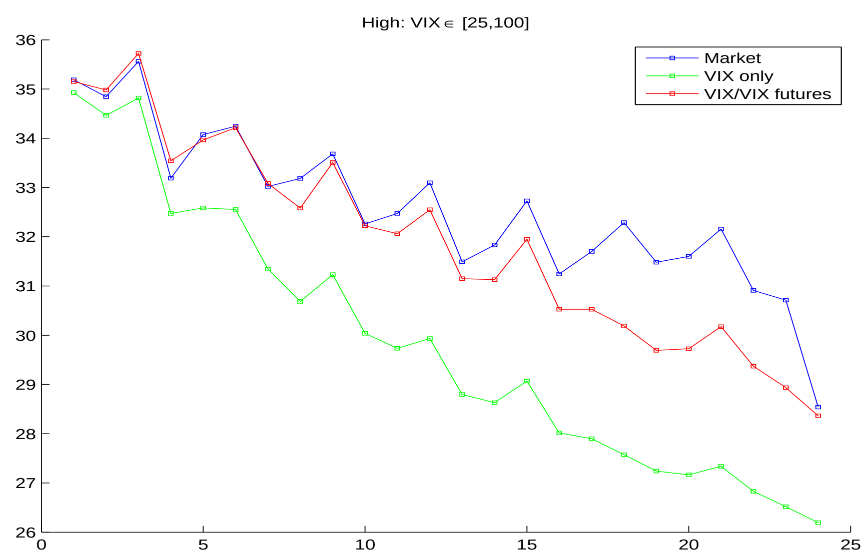
(d) Combination of VIX and VIX futures

FIGURE 1

Model-implied VIX for different methods



(a) Normal VIX level: VIX < 25



(b) High VIX level: VIX ≥ 25

FIGURE 2

Model-implied VIX futures term structures for different VIX levels