Model-free valuation of derivatives

Jim Gatheral

Baruch College
Weissman School of Arts and Sciences
Master of Financial Engineering

National School of Development, Peking University,
Tuesday November 1, 2016

Outline of this talk

- The volatility surface: Stylized facts
- Spanning generalized European payoffs
- The log contract
- Variance swaps
- Weighted variance swaps
- The Bergomi-Guyon expansion
- Robust valuation of weighted variance swaps
- The impact of jumps

Our objective

- Given a stochastic volatility model, no matter how complicated, we can always compute the fair value of derivative assets on the underlying.
  - Though the computation may be very complicated and time-consuming.
- We would like to be able to value derivative securities using only market prices of European options.
  - This turns out to be possible for certain types of derivative claim, notably variance, gamma, and covariance swaps.

The implied volatility smile

- The implied volatility $\sigma_{\text{BS}}(k, \tau)$ of an option (with log-moneyness k and time to expiration τ) is the value of the volatility parameter in the Black-Scholes formula required to match the market price of that option.
- Plotting implied volatility as a function of log-moneyness k generates the *volatility smile*.
- Plotting implied volatility as a function of both k and τ generates the *volatility surface*, explored in detail in, for example, [Gat06].

The SPX volatility surface as of 15-Sep-2005

- We begin with the SPX volatility surface as of the close on September 15, 2005.
  - Next morning is *triple witching* when options and futures set.
- We will plot the volatility smiles, superimposing an SVI fit.
  - SVI stands for "stochastic volatility inspired", a well-known parameterization of the volatility surface.
  - We show in [GJ14] how to fit SVI to the volatility surface in such a way as to guarantee the absence of static arbitrage.
- We then interpolate the resulting SVI smiles to obtain and plot the whole volatility surface.

The March expiry smile as of 15-Sep-2005

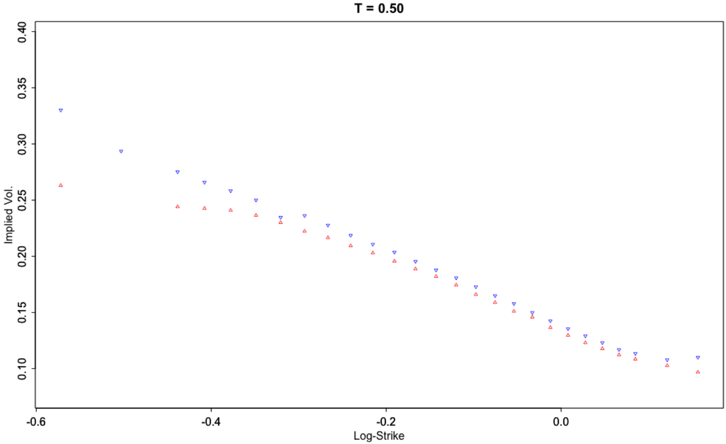

Figure 1: The March expiry smile as of 15-Sep-2005.

SPX volatility smiles as of 15-Sep-2005

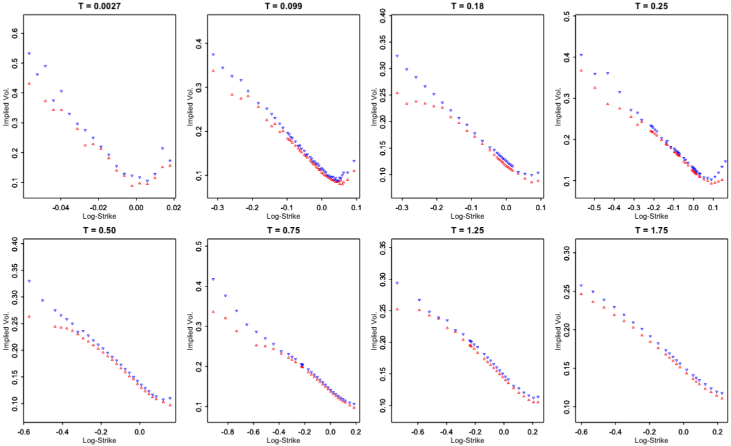

Figure 2: SPX volatility smiles as of 15-Sep-2005.

SPX volatility smiles as of 15-Sep-2005

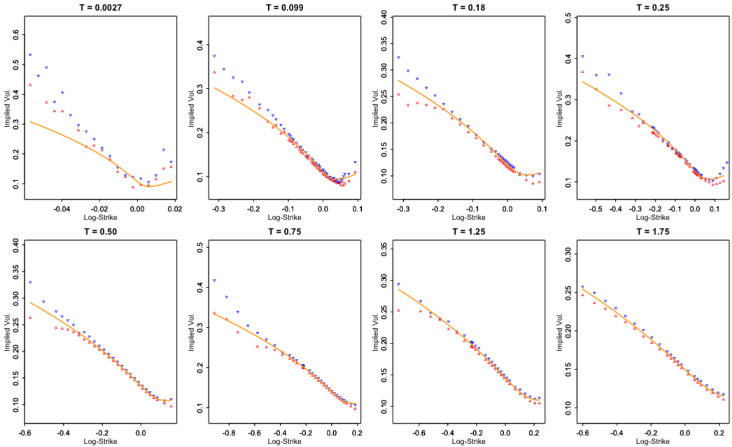

Figure 3: SVI fit superimposed on smiles.

The SPX volatility surface as of 15-Sep-2005

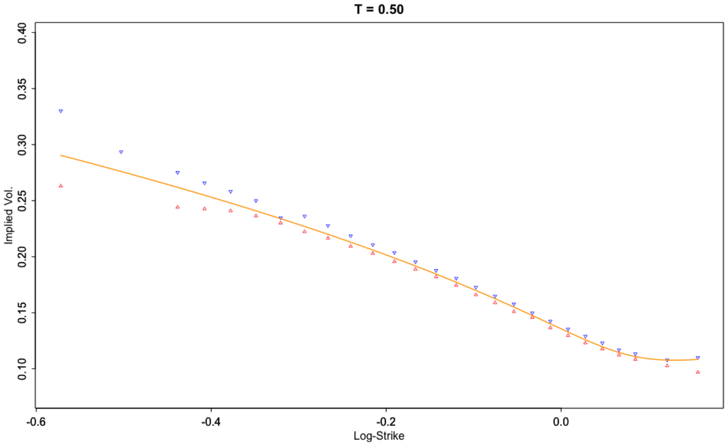

Figure 4: The March expiry smile as of 15-Sep-2005 – the SVI fit looks OK!

The SPX volatility surface as of 15-Sep-2005

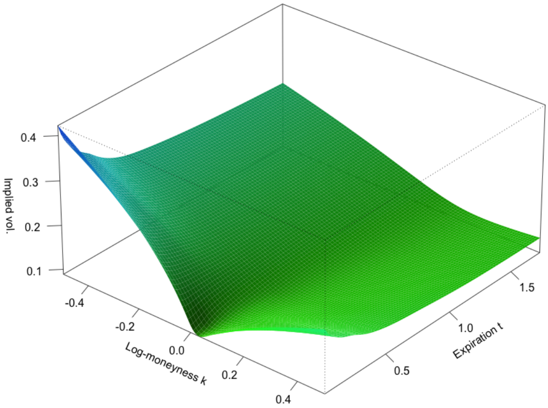

Figure 5: The SPX volatility surface as of 15-Sep-2005 (Figure 3.2 of The Volatility Surface).

Modeling framework

- Having shown that we have market prices for many strikes and expirations, we will now assume that European options with all possible strikes and expirations are traded.
- We will further assume that there are no jumps in the underlying.
  - Though later we will revisit this assumption, estimating the impact of neglecting jumps.

Spanning generalized European payoffs

- We will now show formally that any twice-differentiable payoff at time T may be statically hedged using a portfolio of European options expiring at time T.

Proof from [CM99]

The value of a claim with a generalized payoff $g(S_T)$ at time T is given by

$$\begin{aligned} g(S_T) &= \int_0^\infty g(K)\,\delta(S_T - K)\,dK \\ &= \int_0^F g(K)\,\delta(S_T - K)\,dK + \int_F^\infty g(K)\,\delta(S_T - K)\,dK \end{aligned}$$

Integrating by parts gives

$$\begin{aligned} g(S_T) &= g(F) - \int_0^F g'(K)\,\theta(K - S_T)\,dK \\ &\quad + \int_F^\infty g'(K)\,\theta(S_T - K)\,dK. \end{aligned}$$

... and integrating by parts again gives

$$\begin{aligned} g(S_T) &= \int_0^F g''(K)\,(K-S_T)^+\,dK + \int_F^\infty g''(K)\,(S_T-K)^+\,dK \\ &\quad + g(F) - g'(F)\,\left[(F-S_T)^+ - (S_T-F)^+\right] \\ &= \int_0^F g''(K)\,(K-S_T)^+\,dK + \int_F^\infty g''(K)\,(S_T-K)^+\,dK \\ &\quad + g(F) + g'(F)\,(S_T-F). \end{aligned} \tag{1}$$

Then, with $F = \mathbb{E}[S_T]$,

$$\mathbb{E}\left[g(S_T)\right] = g(F) + \int_0^F dK\,\tilde{P}(K)\,g''(K) + \int_F^\infty dK\,\tilde{C}(K)\,g''(K). \tag{2}$$

- Equation (1) shows how to build any curve using hockey-stick payoffs (if $g(\cdot)$ is twice-differentiable).

Remarks on spanning of European-style payoffs

- From equation (1) we see that any European-style twice-differentiable payoff may be replicated using a portfolio of European options with strikes from 0 to ∞.
  - The weight of each option equal to the second derivative of the payoff at the strike price of the option.
- This portfolio of European options is a static hedge because the weight of an option with a particular strike depends only on the strike price and the form of the payoff function and not on time or the level of the stock price.
- Note further that equation (1) is *completely model-independent*.

Example: European options

- In fact, using Dirac delta-functions, we can extend the above result to payoffs which are not twice-differentiable.
- For example with $g(S_T) = (S_T - L)^+$, $g''(K) = \delta(K - L)$ and equation (2) gives:

$$\begin{aligned}
\mathbb{E}\left[(S_T - L)^+\right] &= (F - L)^+ + \int_0^F dK\,\tilde{P}(K)\,\delta(K - L) \\
&\quad + \int_F^\infty dK\,\tilde{C}(K)\,\delta(K - L) \\
&= \begin{cases} (F - L) + \tilde{P}(L) & \text{if} \quad L < F \\ \tilde{C}(L) & \text{if} \quad L \geq F \end{cases} \\
&= \tilde{C}(L)
\end{aligned}$$

with the last step following from put-call parity as before.

- The replicating portfolio for a European option is just the option itself.

Example: Amortizing options

- A variation on the payoff of the standard European option is given by the amortizing option with strike L with payoff

$$g(S_T) = \frac{(S_T - L)^+}{S_T}.$$

  - Such options look particularly attractive when the volatility of the underlying stock is very high and the price of a standard European option is prohibitive.
- The payoff is effectively that of a European option whose notional amount declines as the option goes in-the-money.
- Then,

$$g''(K) = \left\{ -\frac{2L}{{S_T}^3}\,\theta(S_T - L) + \frac{\delta(S_T - L)}{S_T} \right\}\Big|_{S_T = K}.$$

- Without loss of generality (but to make things easier), suppose $L > F$.
- Substituting into equation (2) gives

$$\begin{aligned}\mathbb{E}\left[\frac{(S_T - L)^+}{S_T}\right] &= \int_F^\infty dK\, \tilde{C}(K)\, g''(K) \\ &= \frac{\tilde{C}(L)}{L} - 2L\int_L^\infty \frac{dK}{K^3}\, \tilde{C}(K)\end{aligned}$$

- We see that an Amortizing call option struck at L is equivalent to a European call option struck at L minus an infinite strip of European call options with strikes from L to ∞.

The log contract

Now consider a contract whose payoff at time T is $\log(S_T/F)$. Then $g''(K) = -\, 1/{S_T}^2\big|_{S_T=K}$ and it follows from equation (2) that

$$\mathbb{E}\left[\log\left(\frac{S_T}{F}\right)\right] \;=\; -\int_0^F \frac{dK}{K^2}\,\tilde{P}(K) - \int_F^\infty \frac{dK}{K^2}\,\tilde{C}(K)$$

Rewriting this equation in terms of the log-strike variable $k := \log\left(K/F\right)$, we get the promising-looking expression

$$\mathbb{E}\left[\log\left(\frac{S_T}{F}\right)\right] \;=\; -\int_{-\infty}^0 dk\, p(k) - \int_0^\infty dk\, c(k) \qquad (3)$$

with

$$c(y) := \frac{\tilde{C}(F e^y)}{F e^y};\; p(y) := \frac{\tilde{P}(F e^y)}{F e^y}$$

representing option prices expressed in terms of percentage of the strike price.

Variance swaps

Assume zero interest rates and dividends. Then $F = S_0$ and applying Itô's Lemma, path-by-path

$$\begin{aligned} \log\left(\frac{S_T}{F}\right) &= \log\left(\frac{S_T}{S_0}\right) \\ &= \int_0^T d\log\left(S_t\right) \\ &= \int_0^T \frac{dS_t}{S_t} - \int_0^T \frac{{\sigma_t}^2}{2} dt \end{aligned} \tag{4}$$

- The second term on the RHS of equation (4) is immediately recognizable as half the total variance (or quadratic variation) $W_T := \langle x \rangle_T$ over the interval $[0, T]$.

- The first term on the RHS represents the payoff of a hedging strategy which involves maintaining a constant dollar amount in stock (if the stock price increases, sell stock; if the stock price decreases, buy stock so as to maintain a constant dollar value of stock).
  - This trivial hedging strategy obviously does not depend on any model.
- Since the log payoff on the LHS can be hedged using a portfolio of European options as noted earlier, it follows that the total variance may be replicated path-by-path in a completely model-independent way so long as the stock price process is a diffusion.
  - In particular, volatility may be stochastic or deterministic and equation (4) still applies.

The log-strip hedge for a variance swap

Now taking the risk-neutral expectation of (4) and comparing with equation (3), we obtain

$$\begin{aligned} \mathbb{E}\left[\int_0^T \sigma_t^2 dt\right] &= -2\,\mathbb{E}\left[\log\left(\frac{S_T}{F}\right)\right] \\ &= 2\left\{\int_{-\infty}^0 dk\, p(k) + \int_0^\infty dk\, c(k)\right\} \end{aligned}$$

- We see that the fair value of total variance is given by the value of an infinite strip of European options in a completely *model-independent* way so long as the underlying process is a diffusion.

Variance swap contracts in practice

- A variance swap is not really a swap at all but a forward contract on the realized annualized variance. The payoff at time T is

$$N \times A \times \left\{ \frac{1}{N} \sum_{i=1}^{N} \left\{ \log\left(\frac{S_i}{S_{i-1}}\right) \right\}^2 - \left\{ \frac{1}{N} \log\left(\frac{S_N}{S_0}\right) \right\}^2 \right\} - N \times K_{var}$$

  where N is the notional amount of the swap, A is the annualization factor and K_{var} is the strike price.
- Annualized variance may or may not be defined as mean-adjusted in practice.

Why variance swaps are beautiful

- From a theoretical perspective, the beauty of a variance swap is that it may be replicated perfectly assuming a diffusion process for the stock price as shown in the previous section.
- From a practical perspective, traders may express views on volatility using variance swaps without having to delta hedge.

History of variance swaps

- Variance swaps took off as a product in the aftermath of the LTCM meltdown in late 1998 when implied stock index volatility levels rose to unprecedented levels.
- Hedge funds took advantage of this by paying variance in swaps (selling the realized volatility at high implied levels).
  - The key to their willingness to pay on a variance swap rather than sell options was that a variance swap is a pure play on realized volatility – no labor-intensive delta hedging or other path dependency is involved.
- Dealers were happy to buy vega at these high levels because they were structurally short vega (in the aggregate) through sales of guaranteed equity-linked investments to retail investors and were getting badly hurt by high implied volatility levels.

Variance swaps in the Heston model

Recall that in the Heston model, the instantaneous variance v satisfies

$$dv_t = -\lambda(v_t - \bar{v})dt + \eta\sqrt{v_t}\,dW_t.$$

Then

$$\begin{aligned}\mathbb{E}\left[\mathcal{V}_0(T)\right] &= \mathbb{E}\left[\int_0^T v_t\,dt\right] \\ &= \frac{1-e^{-\lambda T}}{\lambda}(v-\bar{v}) + \bar{v}\,T. \qquad (5)\end{aligned}$$

The expected annualized variance is given by

$$\frac{1}{T}\mathbb{E}\left[\mathcal{V}_0(T)\right] = \frac{1-e^{-\lambda T}}{\lambda T}(v-\bar{v}) + \bar{v}.$$

Weighted variance swaps

Consider the weighted variance swap with payoff

$$\int_0^T \alpha(S_t)\, v_t\, dt.$$

An application of Itô's Lemma gives the quasi-static hedge:

$$\int_0^T \alpha(S_t)\, v_t\, dt = A(S_T) - A(S_0) - \int_0^T A'(S_u)\, dS_u \qquad (6)$$

with

$$A(x) = 2 \int_1^x dy \int_1^y \frac{\alpha(z)}{z^2}\, dz.$$

- The LHS of (6) is the payoff to be hedged. The last term on the RHS of (6) corresponds to rebalancing in the underlying. The first term on the RHS corresponds to a static position in options given by the spanning formula (1).

Example: Gamma swaps

The payoff of a gamma swap is

$$\frac{1}{S_0} \int_0^T S_t \, v_t \, dt.$$

Thus $\alpha(x) = x$ and

$$A(x) = \frac{2}{S_0} \int_1^x dy \int_1^y \frac{z}{z^2} \, dz = \frac{2}{S_0} \{1 - x + x \log x\}.$$

The static options hedge is the spanning strip for $\frac{2}{S_0} S_t \log S_t$.

- Gamma swaps are marketed as "less dangerous" because higher variances are associated with lower stock prices.

Variance swaps and gamma swaps as traded assets

Denote the time t value of the option strip for a variance swap maturing at T by $\mathcal{V}_t(T)$. That is

$$\mathcal{V}_t(T) = \mathbb{E}\left[\int_t^T v_u\,du \middle| \mathcal{F}_t\right].$$

Similarly, for a gamma swap

$$\mathcal{G}_t(T) = \frac{1}{S_t}\,\mathbb{E}\left[\int_t^T S_u\,v_u\,du \middle| \mathcal{F}_t\right].$$

- Both $\mathcal{V}_t(T)$ and $\mathcal{G}_t(T)$ are random variables representing the prices of traded assets.
    - Specifically, values of portfolios of options appropriately weighted by strike.
        - $\mathcal{V}_t(T)$ is given by the expectation $\mathbb{E}_t[\log S_T]$ of the log contract.
        - $\mathcal{G}_t(T)$ is given by the expectation $\mathbb{E}_t[S_T \log S_T]$ of the entropy contract.

Covariance swaps

Following [Fuk14], consider the covariance swap

$$\mathbb{E}\left[\langle S, \mathcal{V}(T)\rangle_T\right] := \mathbb{E}\left[\int_0^T dS_t\, d\mathcal{V}_t(T)\right].$$

Itô's Lemma gives

$$d(S_t\, \mathcal{V}_t(T)) = S_t\, d\mathcal{V}_t(T) + \mathcal{V}_t(T)\, dS_t + dS_t\, d\mathcal{V}_t(T)$$

so

$$\mathbb{E}\left[\langle S, \mathcal{V}(T)\rangle_T\right] = \mathbb{E}[S_T\, \mathcal{V}_T(T)] - S_0\, \mathcal{V}_0(T) - \mathbb{E}\left[\int_0^T S_t\, d\mathcal{V}_t(T)\right].$$

Noting that $\mathcal{V}_T(T) = 0$ and that $d\mathcal{V}_t(T) = -v_t\,dt$, we obtain

$$\begin{aligned}\mathbb{E}\left[\langle S, \mathcal{V}(T)\rangle_T\right] &= \mathbb{E}\left[\int_0^T S_t\, v_t\, dt\right] - S_0\, \mathcal{V}_0(T) \\ &= S_0\, \left(\mathcal{G}_0(T) - \mathcal{V}_0(T)\right) \\ &=: S_0\, \mathcal{L}_0(T) \qquad (7)\end{aligned}$$

where $\mathcal{L}_0(T) = \mathcal{G}_0(T) - \mathcal{V}_0(T)$ is the *leverage* swap.
Thus the leverage swap gives us the expected quadratic covariation between the underlying and the variance swap.

- This result is completely *model independent* (assuming diffusion), just as in the variance swap and gamma swap cases.

Expression in terms of log and entropy contracts

Going back to the expression of the variance and gamma swaps in terms of log and entropy contracts respectively, we obtain

$$\mathcal{L}_t(T) = \mathcal{G}_t(T) - \mathcal{V}_t(T) = 2\,\mathbb{E}\left[\left(\frac{S_T}{S_t} + 1 \right) \log \frac{S_T}{S_t} \middle| \mathcal{F}_t \right].$$

Heston computations

Heston dynamics are

$$\begin{aligned}\frac{dS_t}{S_t} &= \sqrt{v_t}\,dZ_t \\ dv_t &= -\lambda(v_t - \bar{v})dt + \eta\,\sqrt{v_t}\,dW_t\end{aligned}$$

with $\mathbb{E}\left[dW_t dZ_t\right] = \rho\,dt$. Then

$$d\mathbb{E}\left[S_t\,v_t\right] = -\lambda(\mathbb{E}\left[S_t\,v_t\right] - S_0\,\bar{v})\,dt + \rho\,\eta\,\mathbb{E}\left[S_t\,v_t\right]\,dt.$$

This gives

$$\mathcal{G}_0(T) = \frac{1 - e^{-\lambda' T}}{\lambda' T}\,(v - \bar{v}') + \bar{v}' \tag{8}$$

with

$$\lambda' = \lambda - \rho\,\eta; \quad \bar{v}' = \frac{\lambda}{\lambda'}\,\bar{v}.$$

As before, $\mathcal{L}_0(T) = \mathcal{G}_0(T) - \mathcal{V}_0(T)$.

Forward variance curve formulation

Many (if not most) stochastic volatility models may be recast in the following *forward variance curve* form.

$$\begin{aligned} dx_t &= -\frac{1}{2}\,\xi_t(t)\,dt + \sqrt{\xi_t(t)}\,dZ_t \\ d\xi_t(u) &= \lambda(t,u,\xi_t).dW_t, \quad \xi_0(u) = \xi(u). \end{aligned}$$

$\xi_t(u) = \mathbb{E}\left[\, v_u \middle|\, \mathcal{F}_t \right]$ is the forward variance curve at time t and $Z = \left\{ Z^{(1)}, ..., Z^{(d)} \right\}$ is a $d-$dimensional Brownian motion.

- In particular, the Heston model may be written in this form.
- So can more complicated multi-factor models such as the Bergomi model.

The Bergomi and Guyon expansion

- Using a technique from quantum mechanics, Bergomi and Guyon [BG11] compute an expansion of the volatility smile up to second order in volatility of volatility for stochastic volatility models written in variance curve form.
- The Bergomi-Guyon expansion of implied volatility takes the form

$$\sigma_{\text{BS}}(k,t) = \hat{\sigma}_T + \mathcal{S}_T\, k + \mathcal{C}_T\, k^2 + O(\epsilon^3) \tag{9}$$

Here

$$\begin{aligned}
\hat{\sigma}_T &= \sqrt{\frac{w}{T}}\,\left\{1 + \frac{1}{4\,w}\,C^{x\xi}\right. \\
&\quad \left. + \frac{1}{32\,w^3}\,\left(12\,(C^{x\xi})^2 + w\,(w+4)C^{\xi\xi} + 4\,w\,(w-4)\,C^{\mu}\right)\right\} \\
\mathcal{S}_T &= \sqrt{\frac{w}{T}}\,\left\{\frac{1}{2\,w^2}\,C^{x\,\xi} + \frac{1}{8\,w^3}\,\left(4\,w\,C^{\mu} - 3(C^{x\xi})^2\right)\right\} \qquad (10) \\
\mathcal{C}_T &= \sqrt{\frac{w}{T}}\,\frac{1}{8\,w^4}\,\left(4\,w\,C^{\mu} + w\,C^{\xi\xi} - 6(C^{x\xi})^2\right)
\end{aligned}$$

where $w = \mathcal{V}_0(T) = \int_0^T \xi_0(s)\,ds$ is total variance to expiration T.

Bergomi and Guyon correlation functionals

The various correlation functionals appearing in the BG expansion are:

$$
\begin{aligned}
C^{x\,\xi} &= \int_0^T dt \int_t^T du \, \frac{\mathbb{E}\left[dx_t \, d\xi_t(u)\right]}{dt} \\
C^{\xi\,\xi} &= \int_0^T dt \int_t^T ds \int_t^T du \, \frac{\mathbb{E}\left[d\xi_t(s) \, d\xi_t(u)\right]}{dt} \\
C^{\mu} &= \int_0^T dt \int_t^T du \, \frac{\mathbb{E}\left[dx_t \, dC_t^{x\,\xi}\right]}{dt}.
\end{aligned}
$$

- The Bergomi-Guyon expansion thus gives a one-to-one mapping between ATM level, skew and curvature and model dynamics written in forward variance curve form.

Example: The Heston model

Recall that in the Heston model, v satisfies

$$dv_t = -\lambda(v_t - \bar{v})dt + \eta\sqrt{v_t}\,dW_t.$$

It follows that

$$\xi_t(u) = \mathbb{E}\left[v_u \middle| \mathcal{F}_t\right] = (v_t - \bar{v})\,e^{-\lambda\,(u-t)} + \bar{v}$$

and so

$$d\xi_t(u) = e^{-\lambda\,(u-t)}\,dv_t = e^{-\lambda\,(u-t)}\,\eta\,\sqrt{v_t}\,dW_t.$$

Then

$$\mathbb{E}\left[dx_t\,d\xi_t(u)\right] = \rho\,\eta\,v_t\,e^{-\lambda\,(u-t)}\,dt.$$

Let $w = \mathcal{V}_0(T)$. Then

$$w = \int_0^T \mathbb{E}\left[v_t\right] dt = (v_0 - \bar{v})\,\frac{1 - e^{-\lambda T}}{\lambda} + \bar{v}\,T.$$

Also, with $v_0 = \bar{v}$ to simplify computations, we obtain $w = \bar{v}\,T$ and

$$\begin{aligned} C^{x\xi} &= \rho\,\eta\,\bar{v} \int_0^T dt \int_t^T e^{-\lambda\,(u-t)}\,du \\ &= \frac{\rho\,\eta\,\bar{v}}{\lambda}\left\{1 - \frac{1 - e^{-\lambda T}}{\lambda T}\right\}. \end{aligned}$$

Term structure of ATM skew in the Heston model

Define the at-the-money (ATM) volatility skew

$$\psi(T) = \partial_k \sigma_{BS}(k, T)|_{k=0}$$

and let $w = \mathcal{V}_0(T)$. Then from (10), with $v_0 = \bar{v}$ for simplicity that to first order in η,

$$\begin{aligned}\psi(T) = \mathcal{S}_T &= \sqrt{\frac{w}{T}}\,\frac{1}{2\,w^2}\,C^{x\,\xi} \\ &= \frac{\rho\,\eta}{2\,\sqrt{\bar{v}}}\,\frac{1}{\lambda\,T}\left\{1 - \frac{1 - e^{-\lambda\,T}}{\lambda\,T}\right\}.\end{aligned}$$

- This is consistent with the expression derived in [Gat06] using a different argument.

ATM skew and leverage

To first order in volatility of volatility, the Bergomi-Guyon expansion takes the form

$$\sigma_{\text{BS}}(k,t) = \hat{\sigma}_T + \mathcal{S}_T\, k + O(\epsilon^2) \tag{11}$$

with

$$\begin{aligned} \hat{\sigma}_T &= \sqrt{\frac{w}{T}}\left\{1 + \frac{1}{4\,w}\,C^{x\xi}\right\} \\ \mathcal{S}_T &= \sqrt{\frac{w}{T}}\left\{\frac{1}{2\,w^2}\,C^{x\,\xi}\right\} \end{aligned}$$

where $w = \int_0^T \xi_0(s)\,ds = \mathcal{V}_0(T)$ is total variance to expiration T.

ATM skew and leverage

Moreover, from the definition of $C^{x\xi}$,

$$\begin{aligned} C^{x\,\xi} &= \int_0^T dt \int_t^T du \, \frac{\mathbb{E}\left[dx_t \, d\xi_t(u)\right]}{dt} \\ &= \int_0^T dt \, \frac{\mathbb{E}\left[dx_t \, d\mathcal{V}_t(T)\right]}{dt} \\ &= \mathbb{E}\left[\langle \log S, \mathcal{V}(T)\rangle_T\right] \\ &= \mathbb{E}\left[\langle \log S, \mathcal{G}(T)\rangle_T\right] + \mathcal{O}(\epsilon^2) \\ &= \mathcal{L}_0(T) + \mathcal{O}(\epsilon^2) \end{aligned}$$

where we further used the fact (see [Fuk14]) that $\mathcal{L}_0(T) = \mathbb{E}\left[\langle \log S, \mathcal{G}(T)\rangle_T\right]$.

ATM skew and leverage

Then, squaring (11), we obtain

$$\begin{aligned}\sigma^2_{\mathrm{BS}}(k,t)\,T &= \hat{\sigma}^2_T\,T + 2\,\hat{\sigma}^2_T\,\mathcal{S}_T\,k + O(\epsilon^2)\\ &= w + \frac{C^{x\xi}}{w}\left(k + \frac{w}{2}\right) + O(\epsilon^2)\\ &= \mathcal{V}_0(T) + \frac{\mathcal{L}_0(T)}{\mathcal{V}_0(T)}\left(k + \frac{\mathcal{V}_0(T)}{2}\right) + O(\epsilon^2).\end{aligned}$$

In particular,

$$\psi(T) = \partial_k \sigma^2_{\mathrm{BS}}(k,t)\,T\big|_k = \frac{\mathcal{L}_0(T)}{\mathcal{V}_0(T)} + \mathcal{O}(\epsilon^2).$$

ATM skew and leverage

Thus the leverage swap gives a model-free approximation to the ATM implied volatility skew to first order in volatility of volatility.

- ATM skew and the leverage swap are both related to the covariance between volatility moves and spot moves.
  - In particular, ATM skew and leverage are both zero if spot and volatility moves are uncorrelated.

Robust valuation of swaps

- So far, we have seen that variance, gamma, and covariance swaps may be valued straightforwardly if the prices of Europeans with all possible strikes for a given expiration are known.
  - In practice, we only have a finite number of strike prices listed per expiration.
- One way to estimate the value of such swaps is to fit a parameterization such as SVI, interpolating and extrapolating to fill in all the other strikes.
- We will now show that it is possible to estimate swap values robustly with very little dependence on the interpolation/extrapolation method.

A cool formula

Define

$$d_\pm = -\frac{k}{\sigma_{BS}(k)\sqrt{T}} \pm \frac{\sigma_{BS}(k)\sqrt{T}}{2}$$

and further define the inverse functions $g_\pm(z) = d_\pm^{-1}(z)$. Intuitively, z measures the log-moneyness of an option in implied standard deviations. Then,

$$\mathbb{E}[\mathcal{V}_t(T)] = -2\,\mathbb{E}\left[\log\frac{S_T}{F}\right] = \int_{-\infty}^{\infty} dz\, N'(z)\,\sigma_{BS}^2\left(g_-(z)\right) T \qquad (12)$$

To see this formula is plausible, it is obviously correct in the flat-volatility Black-Scholes case.

Proof

Recall that the fair value of a variance swap under diffusion may be obtained by valuing a contract that pays $2\log\left(S_T/F\right)$ at maturity T. With $w=\sigma_{\mathrm{BS}}^2(k,T)\,T$, brute-force calculation gives

$$
\begin{aligned}
2\mathbb{E}\left[\log\frac{S_T}{F}\right] &= 2\int_0^\infty dK\,\log\left(\frac{K}{F}\right)\,\frac{\partial^2 C}{\partial K^2}\\
&= 2\int_{-\infty}^\infty dk\,k\,N'\left(d_2\right)\left\{-\frac{\partial d_2}{\partial k}\left(1+d_2\frac{\partial\sqrt{w}}{\partial k}\right)+\frac{\partial^2\sqrt{w}}{\partial k^2}\right\}\\
&= 2\int_{-\infty}^\infty dk\,N'\left(d_2\right)\left\{-k\frac{\partial d_2}{\partial k}-\frac{\partial\sqrt{w}}{\partial k}\right\}\\
&= \int_{-\infty}^\infty dk\,N'\left(d_2\right)\,\frac{\partial d_2}{\partial k}\,w
\end{aligned}
$$

which recovers equation (12) as required.

A generalization due to Fukusawa

[Fuk12] derives an expression for the value of a generalized European payoff in terms of implied volatilities.

As one application, he derives the following expression for the value of a gamma swap.

$$\mathbb{E}[\mathcal{G}_t(T)] = 2\,\mathbb{E}\left[\frac{S_T}{F}\log\frac{S_T}{F}\right] = \int_{-\infty}^{\infty} dz\, N'(z)\, \sigma_{BS}^2\,(g_+(z))\, T \quad (13)$$

(note g_+ instead of g_- in the variance swap case).

- In particular, if we have a parameterization of the volatility smile (such as SVI), computing the fair value of the covariance swap is straightforward.

Robust valuation

Following Fukasawa [Fuk12] again, putting $y = N(z)$, we obtain

$$\int_{-\infty}^{\infty} N'(z)\,\sigma^2(z)\,dz = \int_0^1 \sigma^2(y)\,dy.$$

- It turns out that the integrand $\sigma^2(y)$ is typically a very nice function of y in practice.
- The integral is not very dependent on the method of interpolation or extrapolation.

A typical y-integrand

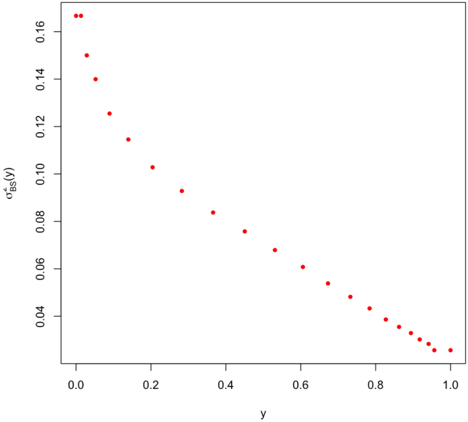

Figure 6: The y-integrand for the Dec-2010 expiration as of 04-Feb-2010. Note the naïve extrapolation.

A Heston experiment

- We consider the volatility surface as of the close on 04-Feb-2010.
- We replace the market prices of options with prices generated from the Heston model with parameters more or less consistent with the volatility surface that day.
    - The strikes and expirations in our dataset are the original market strikes and expirations.
- How close is the robust estimate of the variance swap value to the true value from the closed-form formula?

A fake Heston volatility surface

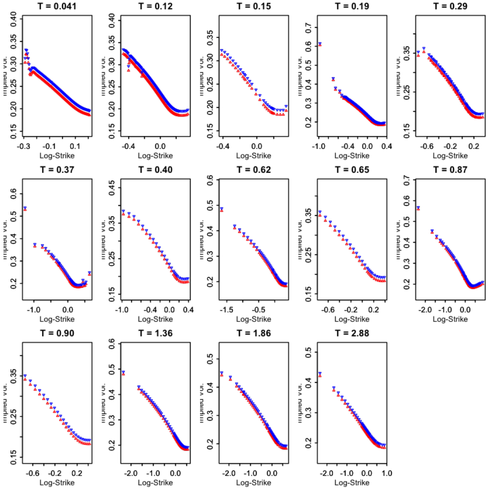

Figure 7: A fake Heston volatility surface based on the market volatility surface as of 04-Feb-2010.

A y-integrand with fake Heston data

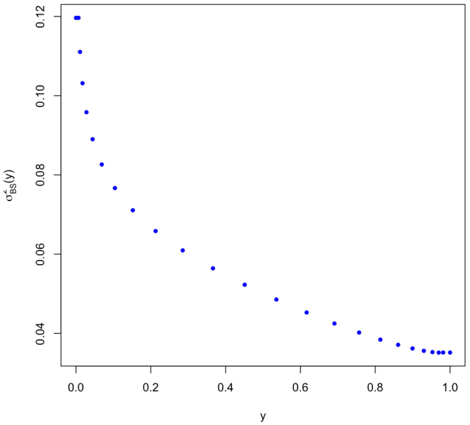

Figure 8: The y-integrand for the Dec-2010 expiration as of 04-Feb-2010 with fake Heston data.

Robust estimates vs exact Heston expressions

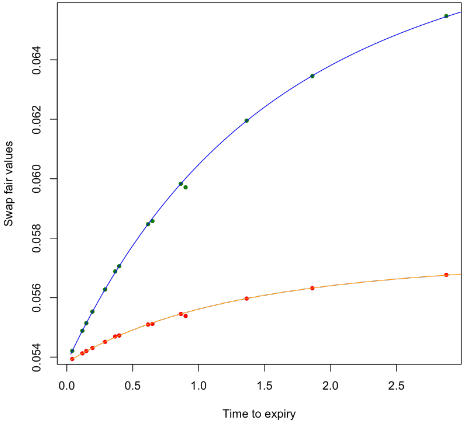

Figure 9: True Heston variance and gamma swap values from (5) and (8) in blue and orange respectively; Fukasawa robust estimates with market strikes and expirations in green and red respectively.

Robust estimates vs exact Heston expressions

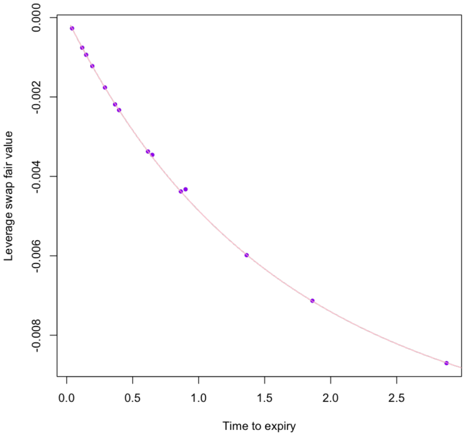

Figure 10: True Heston leverage swap value in pink; Fukasawa robust estimate with market strikes and expirations in purple.

Robust valuation in summary

- Fukasawa's robust valuation method seems to work very well in practice.
  - In particular, rather better than fitting (for example) SVI and performing the integration exactly.

The impact of jumps

- Finally, we examine our assumption that sample paths are continuous.
  - What happens if there are jumps?

Quadratic variation for a compound Poisson process

Let x_t denote the return of a compound Poisson process so that

$$x_T = \sum_{i}^{N_T} y_i$$

with the y_i *iid* and N_T a Poisson process with mean $\lambda\, T$. Define the quadratic variation as

$$\langle x \rangle_T = \sum_{i}^{N_T} |y_i|^2$$

Then

$$\mathbb{E}\left[\langle x \rangle_T\right] = \mathbb{E}\left[N_T\right]\, \mathbb{E}\left[|y_i|^2\right] = \lambda\, T \int_{\mathbb{R}} y^2\, \phi(y)\, dy$$

where $\phi(\cdot)$ is the density of jump sizes.

Also,

$$\mathbb{E}\left[x_T\right] = \lambda\, T \int_{\mathbb{R}} y\, \phi(y)\, dy$$

and

$$\mathbb{E}\left[{x_T}^2\right] = \lambda\, T \int_{\mathbb{R}} y^2\, \phi(y)\, dy + (\lambda\, T)^2 \left(\int_{\mathbb{R}} y\, \phi(y)\, dy\right)^2$$

So

$$\mathbb{E}\left[\langle x\rangle_T\right] = \mathbb{E}\left[{x_T}^2\right] - \mathbb{E}\left[x_T\right]^2 = \mathrm{Var}\left[x_T\right]$$

- Expected quadratic variation is just the variance of the terminal distribution for compound Poisson processes!
    - We know this result is correct for Black-Scholes with constant volatility but obviously it's not true in general (for example in the Heston model).

Option strip for a compound Poisson process

We can express the first two moments of the final distribution in terms of strips of European options using equation (2) as follows:

$$\begin{aligned}
\mathbb{E}\left[x_T\right] = \mathbb{E}\left[\log(S_T/F)\right] &= -\int_{-\infty}^{0} dk\, p(k) - \int_{0}^{\infty} dk\, c(k) \\
\mathbb{E}\left[{x_T}^2\right] = \mathbb{E}\left[\log^2(S_T/F)\right] &= -\int_{-\infty}^{0} dk\, 2\, k\, p(k) - \int_{0}^{\infty} dk\, 2\, k\, c(k)
\end{aligned}$$

- For a compound Poisson process, if we know European option prices, we may compute expected quadratic variation (*i.e.* compute the value of a variance swap) by computing the variance of the terminal distribution.

Compare with diffusion process

On the other hand, if the underlying process is a diffusion, we may compute expected quadratic variation using equation (5) in terms of the log-strip

$$\mathbb{E}\left[\langle x\rangle_T\right] = -2\,\mathbb{E}\left[x_T\right] = 2\left\{\int_{-\infty}^{0} dk\, p(k) + \int_{0}^{\infty} dk\, c(k)\right\}$$

- So, if the underlying process is compound Poisson, we have one way of computing $\mathbb{E}\left[\langle x\rangle_T\right]$ and if the underlying process is a diffusion, we have another.
- In reality, we're not sure what the underlying process is so we would like to know how much difference the choice of underlying process makes.

Computing the difference

To compute the difference, we first note that from the definition of characteristic function,

$$\mathbb{E}\left[\log\left(S_T/F\right)\right] = -i\ \frac{\partial}{\partial u}\phi_T(u)\Big|_{u=0}$$

Also, note that if jumps are independent of the continuous process as they are in both the Merton and SVJ models, the characteristic function may be written as the product of a continuous part and a jump part

$$\phi_T(u) = \phi_T^C(u)\,\phi_T^J(u)$$

where the superscripts C and J refer to the continuous and jump parts respectively.

The Lévy-Khintchine representation

If x_t is a Lévy process, and if the Lévy density $\mu(\xi)$ is suitably well-behaved at the origin, its characteristic function $\phi_T(u) := \mathbb{E}\left[e^{iux_T}\right]$ has the representation

Characteristic function for a Lévy process

$$\phi_T(u) = \exp\left\{i\,u\,\omega\,T - \frac{1}{2}u^2\,\sigma^2\,T + T\int\left[e^{i\,u\,\xi} - 1\right]\mu(\xi)\,d\xi\right\}$$

- ω is set by the Martingale condition $\phi_T(-i) = 1$.
- Explicitly,

$$\omega = \int_{\mathbb{R}}\left(1 - e^{-y}\right)\,\mu(y)\,dy.$$

From the Lévy-Khintchine representation,

$$-i\,\frac{\partial}{\partial u}\,\phi_T^J(u)\Big|_{u=0} = \lambda\,T\int_{\mathbb{R}}\left(1+y-e^y\right)\,\phi(y)\,dy$$

where $\phi(\cdot)$ is the density of jump sizes. On the other hand, we already showed above that

$$\mathbb{E}\left[\langle x^J\rangle_T\right] = \lambda\,T\int_{\mathbb{R}} y^2\,\phi(y)\,dy$$

It follows that the difference between the fair value of a variance swap and the value of the log-strip is given by

$$\mathbb{E}\left[\langle x\rangle_T\right] + 2\,\mathbb{E}\left[x_T\right] = 2\,\lambda\,T\int_{\mathbb{R}}\left(1+y+y^2/2-e^y\right)\,\phi(y)\,dy$$

The effect of jumps is of order jump^3.

- The expression $1 + y + y^2/2$ is just the first three terms in the Taylor expansion of e^y, so the error introduced by valuing a variance swap using the log-strip of equation (5) is of the order of the jump-size cubed.
  - If there are no jumps of course, the log-strip values the variance swap correctly.

Example: lognormally distributed jumps with mean α and standard deviation δ

In this case

$$\begin{aligned}\mathbb{E}\left[\langle x\rangle_T\right]+2\,\mathbb{E}\left[x_T\right] &= \lambda T\left(\alpha^2+\delta^2\right)+2\lambda T\left(1+\alpha-e^{\alpha+\delta^2/2}\right)\\ &= -\frac{1}{3}\lambda T\alpha\left(\alpha^2+3\delta^2\right)+\text{higher order terms}\end{aligned}$$

Putting $\alpha=-0.09$, $\delta=0.14$ and $\lambda=0.61$ (from BCC again), we get an error of only 0.00122427 per year on a one-year variance swap which at 20% vol. corresponds to 0.30% in volatility terms.

An operational definition of diffusion

This analysis allows us to provide an operational definition of diffusion:

- An underlying diffuses (at least approximately) if the third order term in the above Taylor expansion is small.
    - Roughly speaking, this imposes that changes in the underlying between observations should be no greater than 5% or so.
- This is equivalent to saying that Itô's Lemma should provide a good approximation to the change in a function of the process between observations.

Summary

- We showed that weighted variance swaps may be valued independently of any model assuming we know the prices of European options for all strikes for any given expiration and assuming there are no jumps.
  - Path-by-path model-independent replication is also possible.
- We presented the Bergomi-Guyon expansion and derived an approximate relationship between the ATM volatility skew and the leverage (or covariance) swap.
- We then showed that even without all strikes, weighted swaps may be valued robustly with little dependence on the interpolation/extrapolation technique.
- Finally, we show that although the standard variance swap valuation approach assumes diffusion, the existence of reasonably-sized jumps has little effect on their value.

References

Lorenzo Bergomi and Julien Guyon.
The smile in stochastic volatility models.
Available at SSRN 1967470, 2011.

Peter Carr and Dilip Madan.
Option valuation using the fast Fourier transform.
Journal of computational finance, 2(4):61–73, 1999.

Masaaki Fukasawa.
The normalizing transformation of the implied volatility smile.
Mathematical Finance, 22(4):753–762, 2012.

Masaaki Fukasawa.
Volatility derivatives and model-free implied leverage.
International Journal of Theoretical and Applied Finance, 17(01):1450002, 2014.

Jim Gatheral.
The volatility surface: A practitioner's guide.
John Wiley & Sons, 2006.

Jim Gatheral and Antoine Jacquier.
Arbitrage-free SVI volatility surfaces.
Quantitative Finance, 14(1):59–71, 2014.