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February 2025

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# 1 Introduction

Economic theories have successfully characterized and explained historical transitions from the Malthusian stage to modern growth (e.g., Galor and Weil, 2000; Galor, 2011), which follows distinct patterns: Per capita income steadily rises, the population grows despite gradually declining fertility rates, and human capital levels consistently improve. In the past half-century, however, a new pattern has emerged in developed and emerging economies: Economic growth has significantly decelerated; fertility rates have persistently remained below the replacement level, which leads to an aging, stagnating, or even declining population; and only human capital accumulation has continued uninterrupted. No theory has yet characterized these two transitions—from the Malthusian stage to modern economic growth and subsequently to growth slowdown and an aging society—within a unified framework. This paper aims to fill this gap.

Fig.1 illustrates the two transitions in Western Europe—the region where modern economic growth first emerged—by decomposing gross domestic product (GDP) growth into population and per capita income growth. In the early stage (1001-1700 AD), GDP growth was driven almost entirely by population growth, with little per capita income growth, and thus revealed typical characteristics of the Malthusian stage. The first transition occurred in 1700-1820, when per capita income started growing after an extended period of population growth. Economies transitioned from the Malthusian stage to modern growth. Throughout the 19th and 20th centuries, both population and per capita income grew continuously, with population growth rates gradually declining and per capita income growth accelerating. The second transition started around 1970, when the total fertility rate decreased below 2.1 (replacement level), leading to an aging society.[1] Meanwhile, per capita income growth, after peaking between the 1940s and 1960s, began to decline. By the 2000s, the average growth rate had fallen to around 1%, similar to pre-World War II levels. In the 2010s, it declined further, dropping below 0.5%.

Fig.2 further illustrates the second transition in a broader set of economies: OECD Europe, OECD America, the Asian Four (Japan, Singapore, Hong Kong, and South Korea), and BRICS (Brazil, Russia, India, China, and South Africa).[2] For OECD economies and the

[1]Population growth remains slightly positive because life expectancy is increasing.

[2]Historical data on population and per capita income growth during the first transition are less available

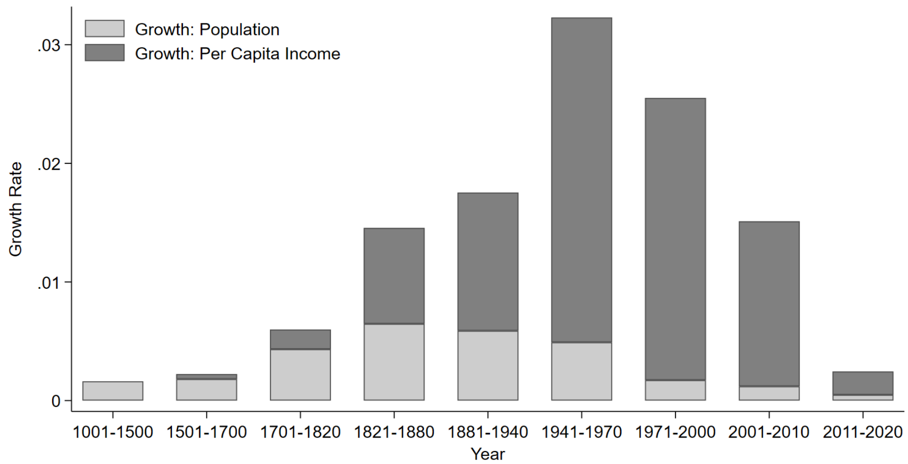


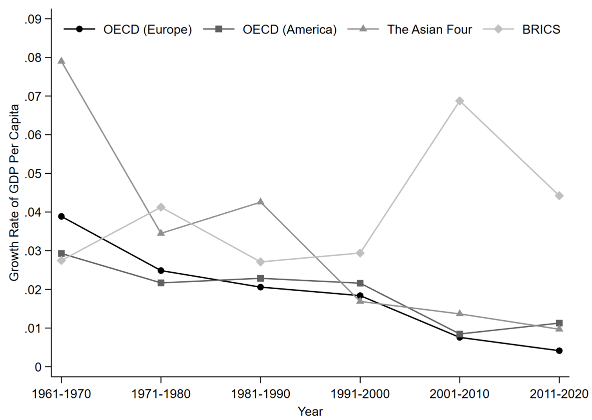

Figure 1: Growth of Population and Per Capita Income in Western Europe

Data Source: Maddison Project Database 2020. Population after WWII excludes international immigrants. Western Europe refers to 19 economies labeled "Western Europe" in Maddison Project Database 2020, aggregated together as a whole. Data on immigrants are from United Nations Population Division (2012) Trends in Total Migrant Stock and United Nations Population Division (2020) International Migrant Stock.

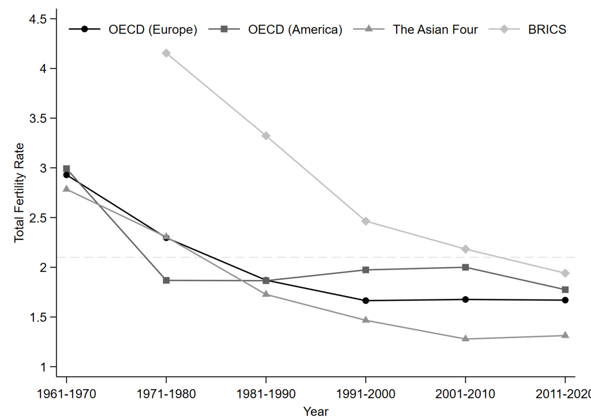
Asian Four, per capita income growth has slowed since the 1960s (Fig.2(a)); the total fertility rate fell in the 1960s and the 1970s and remained below the replacement level afterward (Fig.2(b)), leading to an aging population (Fig.2(c)). Meanwhile, the human capital level continues to grow (Fig.2(d)). BRICS economies, which entered the modern growth stage later than the others, reached their growth peak in the 2000s—a trajectory comparable to earlier experiences of the other economies after World War II. However, like the other economies, BRICS have also seen declining growth rates since the 2000s. Their demographic trends, marked by falling fertility rates, aging populations, and rising human capital levels, now mirror the patterns observed in OECD economies and the Asian Four roughly 30 years earlier.

We develop a model to characterize and explain the two endogenous transitions within a unified framework. Specifically, we integrate endogenous R&D with the child quantity-quality trade-off to explore the long-term relationship between population dynamics and economic growth. In the model, technological progress driven by R&D influences the relative return to human capital compared with the wages of unskilled labor. Anticipating their children's future income, parents adjust their decisions on fertility and investments in

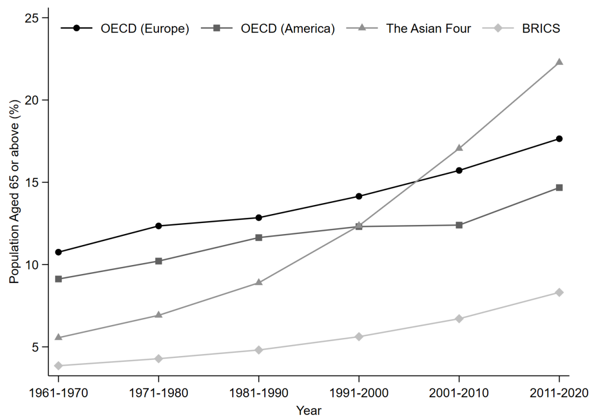
outside Western Europe. We exclude Taiwan due to the unavailability of its data in the World Bank Open Data.



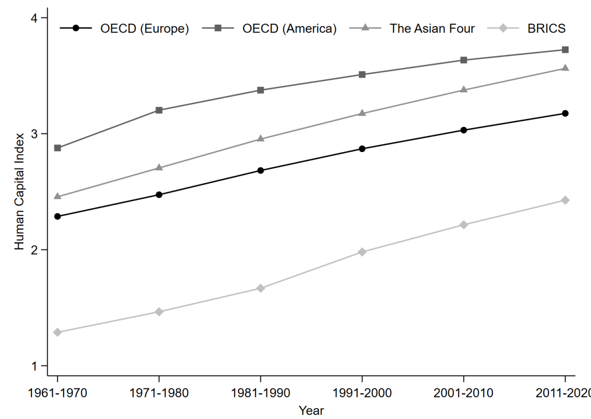

(a) Growth in Per Capita GDP




(b) Total Fertility Rate




(c) Share of Population Aged 65 or above




(d) Human Capital Index

Figure 2: Global Economic Growth and Demographic Transition

Data Source: World Bank Open Data and Penn World Table 10.01. OECD (Europe) refers to 18 initial members of OECD in Europe. OECD (America) refers to Canada and United States. The Asian Four refers to Japan, Singapore, Hong Kong, and South Korea. BRICS refers to Brazil, Russia, India, China and South Africa. Total fertility rate and human capital index are, respectively, the average total fertility rate and human capital index of each economy, weighted according to population size. Population aged 65 or above and the growth rate of GDP per capita are calculated by aggregating all economies in the group.

child human capital based on their expectation of this relative return. These decisions, in turn, shape future population growth and the supply of human capital, which subsequently affect the costs and benefits of R&D. The model describes the economy's transition through four distinct phases: (i) the Malthusian phase, characterized by high fertility and stagnant technological progress, resulting in growth in population without growth in per capita income; (ii) the prosperous phase, during which technological progress accelerates along with population growth, declining fertility rates, and rising investment in human capital; (iii) the aging phase, marked by low fertility, a shrinking and aging population, and decelerating technological progress and economic growth, despite ongoing human capital accumulation; and (iv) the stagnation phase, in which sustained population decline eventually halts both R&D and economic growth and causes the economy to enter a steady state.

We start the analysis in Section 2 by setting the environment for our model, which integrates endogenous R&D (Romer, 1990) with the child quantity-quality trade-off (Becker and Barro, 1988). The economy, which consists of households, producers, and developers of new technology, features overlapping generations and discrete time. Each single-person household lives through three life stages: childhood, young adulthood, and old adulthood. During childhood, individuals receive human capital investment from their parent. In young adulthood, they make decisions regarding consumption, allocate their time between the labor market, fertility, and human capital investment for their children, and save through investments in fixed resources. In old adulthood, they consume all savings and returns from fixed resource investments. The economy has two production sectors: traditional and modern. In the traditional sector, firms employ unskilled labor and fixed resources to produce goods traded in a competitive market. In the modern sector, firms use human capital exclusively to produce a variety of modern goods. Newly developed modern goods are protected by patents and sold monopolistically for one period, while previously developed goods are traded competitively. Firms choose whether to engage in R&D, which uses human capital only to innovate new modern goods, based on the monopoly profits of new goods and the cost of human capital.

For the first time in the literature, our model setup bridges the child quantity-quality trade-off and the endogenous R&D choices through the supply and demand for human capital in equilibrium. First, the child quantity-quality trade-off is determined by parents' expected relative return to human capital versus wages for unskilled labor. Second, the size of the R&D sector is determined by the relative profit of a new good versus the cost of human capital.

In Section 3, we characterize the equilibrium. It features two potential corner solutions, which determine the transition through four phases. The first concerns whether parents invest in their children's human capital. The model demonstrates that zero investment occurs only in the absence of modern technology and when the population is small. As the population grows, the expanding market size makes R&D in modern technology profitable, which induces the demand for human capital and enables the economy to move out of the Malthusian phase. The second corner solution pertains to firms' decisions to engage in R&D. The model shows that technological stagnation can result from competition for human capital between the production and R&D sectors. As technology advances, the demand for human capital in the production sector increases, which diminishes the profitability of R&D. Since the supply of human capital is closely tied to population size, the model concludes that excessive population decline leads to technological stagnation. Finally, we derive the price system, prove the existence and uniqueness of the equilibrium, and show that the economy's dynamics can be fully described by two state variables: population and the technological frontier.

We further analyze the dynamics of the economy in Section 4. We define two critical loci of the state variables (population against the technological frontier): One indicates a fertility rate at replacement level and the other zero R&D investment. By examining the properties of these loci, the model demonstrates that the economy endogenously transitions through four distinct phases. Starting with a small population without modern technology, the economy goes through the Malthusian phase, characterized by population expansion without human capital investment or R&D. Once the population exceeds a threshold, the economy enters the prosperous phase, marked by accelerating technological progress, continued population growth, declining fertility rates, and rising human capital investment. As fertility falls below the replacement level, the aging phase emerges, characterized by population decline, an increasing proportion of elderly individuals, and slower technological progress and economic growth. Finally, when the population drops below a critical level the stagnating phase occurs, which features both population decline and technological stagnation. The economy ultimately converges to a steady state with stagnant technology and a stabilized population.

The unique predictions of our model arise from a market failure in pricing fertility. In equilibrium, the relative return to human capital merely captures the scarcity of human capital relative to unskilled labor in the production and R&D sectors; this relative return is then used by the representative and individual parent as a price signal to determine the trade-off between fertility and investment in child human capital. Consequently, parents fail to internalize the positive externalities of fertility at societal level beyond just providing unskilled labor. These externalities include expanding market size, increasing the total supply of human capital, and sustaining future population growth in the model. As a result, technological progress leads to population decline and, ultimately, economic stagnation.

In Section 5, we examine the central role of the child quantity-quality trade-off, which links two fundamentally endogenous choices in the economy: fertility and R&D. On one hand, firms' R&D decisions generate the demand for human capital, which in turn affects parental fertility choices. On the other, parental fertility decisions impact the supply of human capital, which then shapes firms' R&D decisions. In equilibrium, population dynamics and R&D are simultaneously determined. To illustrate this point, we analyze the dynamics of our model by considering an exogenous birth policy that fixes the fertility rate and thus shutting down the child quantity-quality trade-off . In this scenario, technological stagnation is temporary, while technology and per capita income grow persistently; this leads to a balanced growth path similar to the result in Jones (1995a). This exercise confirms that the unique predictions of our model—such as the four-phase transition, inevitable aging, and stagnation—are not driven by specific assumptions about production or R&D but rather by the child quantity-quality trade-off.

Given the centrality of this trade-off, we conclude by discussing potential paths to achieve sustainable economic growth. Pronatalist policies can help parents internalize the externalities. Also, breakthroughs in human capital or knowledge production technologies could change the relationship between demographic dynamics and technological progress, and thereby support permanent growth.

## 1.1 Related Literature

Our study bridges two strands of the literature on economic growth. One seeks to provide a micro-foundation to endogenize fertility and derives the implication for economic growth. In these models, parents are assumed to be either altruistic and care about the number, income, or utility of their children;[3] or selfish and care about the transfers from their children when the parents are old.[4] Considering the close relationship between human capital and economic growth, Becker et al. (1990) introduces the trade-off between fertility (quantity)

[3]Models of parental utility from children's utility include Becker and Barro (1988) and Barro and Becker (1989); from numbers, Galor (2012) and Baudin et al. (2015); and from potential income, Galor and Weil (2000), Greenwood and Seshadri (2002), and Galor and Mountford (2008).

[4]These models include Morand (1999); Becker et al. (2016).

and human capital per child (quality) to the growth model.[5] Introduction of the quantity-quality trade-off greatly enriches the growth theory (e.g., Greenwood and Seshadri, 2002; De la Croix and Doepke, 2003; De la Croix, 2013; Guo et al., 2022; Baudin and De la Croix, 2024). In particular, given that the return to human capital depends endogenously on technological development, unified growth models—such as those by Galor and Weil (2000); Tamura (2002); Doepke (2004); and Galor and Mountford (2008)—explicitly link population dynamics with technological progress.[6]

Despite the crucial role of the return to human capital in this literature, it does not provide a micro-foundation for technological progress, such as R&D choices, that drives the demand for human capital. Typically, this literature derives the demand for human capital directly from a specific function for technological progress, without deriving it from the optimization decision of potential technological developers. Consequently, model predictions are sensitive to functional form assumptions on technological progress.

Importantly, the other strand of the literature has sought to build a micro-foundation for technological progress for decades, such as the endogenous R&D models. The predictions of these R&D models hinge crucially on two assumptions: One concerns the functional forms for R&D and the other the population dynamics. By assuming that R&D exhibits a "scale effect"—whereby the growth rate of technology is proportional to the total resources devoted to R&D—early endogenous R&D models predict permanent economic growth as long as the population remains constant or increases (Romer, 1990; Grossman and Helpman, 1993; Aghion and Howitt, 1992). Relaxing the assumption on the "scale effect", some late studies predict permanent economic growth only when the population grows continuously (Jones, 1995a; Kortum, 1997; Segerstrom, 1998).[7] Despite different predictions, both frameworks assume exogenous non-negative population growth.

In contrast to this model assumption, fertility rates have fallen below replacement levels and the population has shrunk in many developed and emerging countries. Recently, Jones

[5]Introduction of the child quantity-quality trade-off also seeks to explain the negative income elasticity of fertility in the post-Malthusian era (Becker and Lewis, 1973; Doepke, 2015).

[6]Some other models, such as Goodfriend and McDermott (1995) and Hansen and Prescott (2002), link technological progress to human capital or population without relying on the quantity-quality trade-off.

[7]Jones (1995b) challenges the "scale effect" with empirical evidence. Jones (1995a) therefore proposes that the growth level of technology, rather than the growth rate, depends on R&D resources. Other notable models that eliminate the "scale effect" include Dinopoulos and Thompson (1998) and Howitt (1999).

(2022) examines this issue. With an exogenous decline in population, endogenous growth models with or without the "scale effect" predict a stagnating steady state in both technology and per capita income, while population eventually declines to zero—an outcome he refers to as the "empty planet" result. When fertility is endogenized, multiple steady states often emerge. Which state is realized depends on the values of initial state variables and specific parameter values related to fertility costs, parental time preferences, the degree of parental altruism toward children, and R&D technology.

Our study connects these two strands of literature and establishes two micro-foundations for both the child quantity-quality trade-off and endogenous R&D. The connection of these two micro-foundations yields three benefits. First, our model generates rich dynamics in both population and technology, capturing the two endogenous transitions within a unified framework. Specifically, the population endogenously shifts from fast growth to decelerated growth, and subsequently to an aging society; the technology endogenously shifts from pre-modern stagnation to modern growth, and subsequently to slowdown; also, human capital investment shifts from almost non-existence to sustainable growth. All of these predicted dynamics are consistent with the stylized facts presented in Figs.1-2. Second, our model offers a new explanation for the emergence of fertility rates below the replacement level. In our model, this is an endogenous result of technological progress, which increases the relative returns to human capital. In turn, this incentivizes parents to allocate more resources toward improving the human capital per child (quality) rather than increasing fertility (quantity). Third, our model predictions depend less on the assumptions of functional form for R&D, such as the presence or absence of the "scale effect", and specific values for initial state variables and parameters.

Our model differs from Jones (2022) by predicting a fertility rate below the replacement level and an aging population as inevitable endogenous outcomes. In Jones (2022), when fertility is endogenized, a fertility rate below the replacement level is one of multiple possible steady states, contingent on parameter values and initial conditions. In contrast, our model demonstrates that technological progress inevitably leads to persistently low fertility. The robustness of this result is largely independent of specific parameter values, as it emerges from the interplay between parental decisions on the quantity-quality trade-off and firms' decisions for R&D investment. By contrast, Jones (2022) does not consider the child quantity-quality trade-off.

## 2 Environment

This section presents a framework that integrates the child's quantity-quality trade-off (Becker and Barro, 1988) and endogenous R&D (Romer, 1990). The economy, consisting of households, producers, and developers of new technology, features overlapping generations and discrete time.

### 2.1 Household

We model a couple as a single individual who lives through three periods. In the first period, as a child, she acquires human capital through her parents' time investment. In the second period, as a young adult, she makes decisions regarding her (1) consumption of traditional and modern goods; (2) fertility; (3) time allocation between working and investing in her children's human capital; and (4) financial investment in fixed resources. Her decisions on fertility and time investment in children are motivated by pure altruism. In the final period, as an old adult, she consumes returns from her fixed resource investment, with no intergenerational transfers assumed.

#### 2.1.1 Preference, Budget Constraints, and Human Capital Production

For a young adult in period $t$, her utility is

$$u_t = v\left(c^y_{T,t}, \{c^y_{M,t}(z)\}_0^{\bar{z}_t}\right) + \beta \cdot v\left(c^o_{T,t+1}, \{c^o_{M,t+1}(z)\}_0^{\bar{z}_{t+1}}\right) + \beta\gamma \cdot \ln\left[n_t\left(w_{t+1} + r_{t+1}h_{t+1}\right)\right], \quad (1)$$

where

$$v\left(c_T, \{c_M(z)\}_0^{\bar{z}}\right) = \frac{1}{\rho}\ln\left((c_T)^\rho + \int_0^{\bar{z}} (c_M(z))^\rho \,\mathrm{d}z\right);\ \rho \in (0,1).$$

In this setup, $c^y_{T,t}$ and $c^y_{M,t}(z)$ represent her consumption of traditional and modern goods in period $t$, where $z$ denotes the variety of modern goods and $\bar{z}_t$ is the domestic technological frontier. $c^o_{T,t+1}$ and $c^o_{M,t+1}(z)$ denote her consumption in period $t+1$ when she is old. $n_t$ is her fertility, and $h_{t+1}$ the human capital for each of her children. $w_{t+1}$ and $r_{t+1}$ are, respectively, the wage for unskilled labor and the return to human capital in period $t$+1. The parameters $\beta$ and $\gamma$ represent the discount factor and the degree of altruism, while $\rho$ governs the elasticity of substitution between different goods.

In Eq.(1), we adopt the framework of Galor and Weil (2000) and Galor and Mountford (2008), in which the parent is concerned with the total income of her children, rather than the approach by Becker et al. (1990), which focuses on her children's utilities. Our results remain consistent under the latter framework, since the parent in our model influences her children's utility solely through time investment in their human capital.

Her budget constraint in period $t$ is given by

$$c^y_{T,t} + \int_0^{\bar{z}_t} P_t(z) c^y_{M,t}(z) \mathrm{d}z + q_t x_{t+1} = (1 - n_t(\tau + e_t)) (w_t + r_t h_t), \tag{2}$$

where $P_t(z)$ and $q_t$ are, respectively, the price of modern good $z$ and the price of fixed resources in period $t$, and $x_{t+1}$ denotes her fixed resource investment for period $t+1$,[8] $\tau$ captures the fixed time cost of raising a child, and $e_t$ represents the time invested in each child's human capital. The price of the traditional good is normalized to 1. In young adulthood, the individual is endowed with one unit of time, which she allocates between household activities and the labor market. Her labor income, consisting of the returns from unskilled labor ($w_t$) and human capital ($r_t h_t$), is divided between current consumption and investment in fixed resources.

Her budget constraint in period $t+1$, when she reaches old age, is given by

$$c^o_{T,t+1} + \int_0^{\bar{z}_{t+1}} P_{t+1}(z) c^o_{M,t+1}(z)\, dz = (d_{t+1} + q_{t+1}) x_{t+1}, \tag{3}$$

where $d_{t+1}$ represents the return on her fixed resources in period $t+1$. In old age, she no longer participates in the labor market. Instead, she rents out her fixed resources to earn a return, and eventually sells these resources to the next generation.

The human capital production function, $\eta : [0, +\infty) \to [0, +\infty)$, is

$$h_{t+1} = \eta(e_t). \tag{4}$$

[8]Fixed resources refer to production factors that remain constant for the entire economy in each period, such as land.

We assume the following properties for $\eta(\cdot)$:

**Assumption 1.** *The human capital production function $\eta$ is twice continuously differentiable ($C^2$), strictly increasing, and strictly concave. That is, for all $e \in [0, +\infty)$, $0 < \eta'(e) < +\infty$ and $\eta''(e) < 0$. Also, if no time is invested, the human capital level is zero, i.e., $\eta(0) = 0$.*

### 2.1.2 Optimization

In period $t$, the young adult maximizes her utility (Eq.(1)) by choosing her consumption, financial investment in fixed resources, fertility, and time investment in her children's human capital, subject to the budget constraints (Eqs. (2)-(3)), the human capital production function (Eq.(4)), and the non-negativity of all choice variables.

Since the utility function is concave and the budget set is convex, Kuhn-Tucker conditions are sufficient to solve the optimization problem. Rearranging these conditions, we first observe

$$n_t(\tau + e_t) = \frac{\beta\gamma}{1 + \beta + \beta\gamma},$$

which indicates that the parent allocates a portion of her time to rearing child, which is further divided between childbirth and investment in the child's human capital. This explicitly reflects the trade-off between having more children (quantity) and investing in their human capital (quality).

We then find

$$\eta'(e_t)(\tau + e_t) - \eta(e_t) + \left[\frac{\tau}{\beta\gamma}\lambda_t - 1\right]\frac{w_{t+1}}{r_{t+1}} = 0, \ \lambda_t \geq 0, \ e_t \geq 0, \ \lambda_t e_t = 0,$$

where $\lambda_t$ is the Lagrangian multiplier for the non-negativity constraint on $e_t$. From this, we derive the following lemma.

**Lemma 1.** *Under Assumption 1, the optimal parental time investment in children's human capital is uniquely determined by the future relative return of human capital $r_{t+1}/w_{t+1} \equiv s_{t+1}$:*

$$e_t = \varepsilon\left(s_{t+1}\right),$$

*where $\varepsilon$ is a continuous function such that if $s \leq [\tau\eta'(0)]^{-1}$, $\varepsilon(s) = 0$; if $s > [\tau\eta'(0)]^{-1}$,*

$\varepsilon(s) > 0$ *and* $\varepsilon'(s) > 0$.[9]

Lemma 1 states that parental time investment in children's human capital is determined by the future relative return of human capital to the wage of unskilled labor. If the relative return is below a threshold, parents invest no time in human capital and focus entirely on childbirth, reflecting a Malthusian phase of the economy. If the relative return exceeds the threshold, parents allocate time to investing in children's human capital, with investment increasing with the relative return.

*Remark.* This result is consistent with the literature. Galor and Weil (2000) theoretically derive that parents positively invest children's human capital only if technological progress is sufficiently rapid, assuming that the rate of technological progress directly influences the marginal productivity of parental time investment in human capital production. Galor and Mountford (2008) refine this assumption by proposing that technological progress affects children's human capital through the change in wages. They demonstrate that parents are more likely to raise skilled children when the wage gap between skilled and unskilled workers reaches a certain level. We extend this line of research by allowing for a continuous investment in child human capital.

Finally, we derive a trade-off between the consumption of traditional and modern goods,

$$c^i_{M,s}(z) = P_s(z)^{-\frac{1}{1-\rho}} c^i_{T,s}, \ \forall z \in [0, \bar{z}_s], \ (s,i) \in \{(t,y),(t+1,o)\}, \tag{5}$$

and a trade-off between consumption in periods $t$ and $t+1$,

$$q_t x_{t+1} = \frac{\beta}{1+\beta+\beta\gamma}(w_t + r_t h_t).$$

## 2.2 Production

The production sector consists of representative traditional goods producers and a continuum of modern goods producers. The market for traditional goods is perfectly competitive, while the market structure for modern goods depends on whether the good is newly developed.

[9] Proofs for all lemmas and propositions are provided in Appendix A.

### 2.2.1 Production of Traditional Goods

Traditional goods are produced using fixed resources and unskilled labor with a Cobb-Douglas technology. In a completely competitive market, a representative traditional producer in period $t$ maximizes its profit:

$$\max_{X_t^p, L_t^p} \; A(X_t^p)^{1-\alpha}(L_t^p)^{\alpha} - d_t X_t^p - w_t L_t^p; \;\; \alpha \in (0,1),$$

where $X_t^p$ and $L_t^p$ are, respectively, the employed fixed resource and unskilled labor, and $A$ is the total factor productivity. By solving the problem, we have

$$w_t = A\left(\frac{X_t^p}{L_t^p}\right)^{1-\alpha}; \; d_t = A\left(\frac{L_t^p}{X_t^p}\right)^{\alpha}.$$

### 2.2.2 Production of Modern Goods

Modern goods are manufactured using human capital with a linear technology,[10] with productivity normalized to 1. For all pre-developed goods $z \in [0, \bar{z}_{t-1}]$, producers operate in a completely competitive market. They maximize their profit:

$$\max_{H_t^p(z)} \; P_t(z)H_t^p(z) - r_t H_t^p(z), \tag{6}$$

where $H_t^p(z)$ is the human capital employed to produce good $z$.

For any newly developed good $z \in (\bar{z}_{t-1}, \bar{z}_t]$, a one-period patent is granted to the developer. It operates as a monopolist in this period by

$$\max_{H_t^p(z), P_t(z)} \; \pi_t(z) = P_t(z)H_t^p(z) - r_t H_t^p(z), \tag{7}$$

subject to the demand for modern goods, which is derived by aggregating Eq.(5) across all consumers:

$$H_t^p(z) = P_t(z)^{-\frac{1}{1-\rho}}\left(L_t c_{T,t}^y + L_{t-1} c_{T,t}^o\right), \tag{8}$$

where $\pi_t(z)$ is the profit of the newly developed good $z$ and $L_t$ $(L_{t-1})$ is the population of

[10] We make this simplification without loss of generality, as the concavity of the production function of modern goods can be captured by the concavity of the production function of human capital ($\eta$).

the young (old) cohort in period $t$. Note that $\pi_t(z) = \pi_t$; that is, the monopoly profit is identical among all the newly developed goods, as the solution of problem (7) subject to (8) is independent of $z$. This is a standard result in the literature on endogenous R&D models (e.g., Romer, 1990).

By solving problems (6) and (7), we have

$$P_t(z) = \begin{cases} r_t, \text{ if } z \in [0, \bar{z}_{t-1}], \\ \dfrac{r_t}{\rho}, \text{ if } z \in (\bar{z}_{t-1}, \bar{z}]. \end{cases}$$

## 2.3 R&D

Prospective technology developers decide whether to participate in R&D each period. Developers employ human capital to develop new goods, striving to outpace competitors and secure patents. The probability of obtaining a successful patent depends on the previous technological frontier and the number of competitors. Once a new good is developed, the developer earns the monopoly profit for one period.

In period $t$, the measure of varieties of new goods is $\Delta(\bar{z}_{t-1}, N_t) > 0,\ \forall \bar{z}_{t-1} \geq 0,\ N_t > 0$; and $\Delta(\bar{z}_{t-1}, 0) = 0$, $\forall \bar{z}_{t-1} \geq 0$. Here, $N_t = \int_0^\infty m_{it} \mathrm{d}i$ is the measure of developers engaged in R&D in period $t$, where $m_{it} = 1$ if developer $i$ participates in R&D and $m_{it} = 0$ otherwise. The probability of a successful patent for each developer is $\Delta(\bar{z}_{t-1}, N_t)/N_t \equiv \delta(\bar{z}_{t-1}, N_t)$. Without competitors, this probability is $\delta(\bar{z}_{t-1}, 0) = \lim_{N \to 0} \Delta(\bar{z}_{t-1}, N)/N \equiv \delta_0(\bar{z}_{t-1})$. The demand for human capital in each R&D program is $\omega(\bar{z}_{t-1}) > 0$. A potential developer's decision problem is

$$\max_{m_{it} \in \{0,1\}} \ m_{it} \left[ \delta(\bar{z}_{t-1}, N_t)\pi_t - \omega(\bar{z}_{t-1}) r_t \right]. \tag{9}$$

Several regularity conditions of $\Delta$ and $\omega$ are given as follows.

**Assumption 2.** $\Delta$ *and* $\omega$ *are both continuously differentiable* ($C^1$) *and for* $\forall \bar{z} \geq 0$ *satisfy:* i. $\Delta_2(\cdot,\cdot) > 0$. ii. $\delta_2(\cdot,\cdot) < 0$. iii. $\lim_{N\to 0} \delta(\bar{z}, N) > 0$, $\lim_{N \to \infty} \delta(\bar{z}, N) = 0$. iv. $\mathrm{d}[\omega(\bar{z})/\delta_0(\bar{z})]/\mathrm{d}\bar{z} \geq 0$.

Assumptions 2.i-iii are straightforward. In Assumption 2.iv, $\omega(\bar{z})/\delta_0(\bar{z})$ represents the expected human capital inputs for successful R&D without competitors. The assumption specifies that the expected inputs do not decrease as the technological frontier advances.

The expected inputs are analogous to the average cost of R&D in the endogenous R&D literature. Romer (1990), Grossman and Helpman (1993), and Aghion and Howitt (1992) assume constant average costs, while Kortum (1997) and Segerstrom (1998) derive increasing average costs from certain micro-foundations. Assumption 2.iv is consistent with these studies.[11]

*Remark.* The relationship between R&D ($\Delta$), the previous technological frontier ($\bar{z}_{t-1}$), and total human capital devoted to R&D ($N_t\omega(\bar{z}_{t-1})$) is widely discussed in the literature. Early endogenous growth models (e.g., Romer (1990); Grossman and Helpman (1993); and Aghion and Howitt (1992)), despite differing micro-foundations, consistently exhibit a "scale effect" at the aggregate level (Jones, 1995a), which indicates that the technological growth rate is proportional to total R&D resources. Using our notation, this is expressed as $\Delta/\bar{z}_{t-1} = f(N_t\omega(\bar{z}_{t-1}))$, which implies a linear relationship between R&D ($\Delta$) and the previous technological frontier ($\bar{z}_{t-1}$), given the total resources ($N_t\omega(\bar{z}_{t-1})$). However, empirical studies hardly find supportive evidence for this "scale effect" (e.g. Jones, 1995b). Correspondingly, later studies, such as Kortum (1997); Segerstrom (1998); and Jones (2002), propose that the growth rate of technology decreases with the previous technological frontier, given the total resources. For example, Jones (2002) specifies (using our notation) $\Delta = f(N_t\omega(\bar{z}_{t-1}))\bar{z}_{t-1}^{\psi}$, $\psi < 1$. Assumption 2 imposes little restriction in this regard, which means that our results below do not depend on any specific relationship between the three factors at the aggregate level.

**Lemma 2.** *Under Assumption 2, the measure of developers who participate in R&D is uniquely determined by the monopoly profit, the return of human capital, and the current technological frontier:*

$$N_t = \Phi\left(\frac{\pi_t}{r_t}, \bar{z}_{t-1}\right),$$

*where* $\Phi$ *is a continuous function. Furthermore,* $\Phi(\pi_t/r_t, \bar{z}_{t-1}) > 0$ *and* $\Phi_1(\pi_t/r_t, \bar{z}_{t-1}) > 0$ *iff*

$$\frac{\pi_t}{r_t} > \frac{\omega(\bar{z}_{t-1})}{\delta_0(\bar{z}_{t-1})}$$

[11]Our main results below remain robust if we adopt a weaker assumption than Assumption 2.iv, specifically: $\lim_{\bar{z}\to\infty} \frac{\omega(\bar{z})}{\delta_0(\bar{z})} > 0$. This implies that no matter how advanced technology becomes, the human capital required for R&D cannot approach zero indefinitely.

*and* $\Phi(\pi_t/r_t, \bar{z}_{t-1}) = 0$ *otherwise.*

Lemma 2 states that the relative profit of a new good to the cost of human capital ($\pi_t/r_t$) determines the size of the R&D sector. If the relative profit falls below the expected human capital inputs for a successful R&D ($\omega/\delta_0$), no developers will engage in R&D, which results in technological stagnation. When the relative profit exceeds the expected inputs, the R&D sector emerges and expands as the relative profit rises.

# 3 Equilibrium

This section defines and analyzes the equilibrium of the economy. We start by clarifying market-clearing conditions and the laws of motion. In period $t$, the market-clearing conditions for traditional and modern goods are

$$L_t c^y_{T,t} + L_{t-1} c^o_{T,t} = A(X^p_t)^{1-\alpha}(L^p_t)^{\alpha},$$
$$L_t c^y_{M,t}(z) + L_{t-1} c^o_{M,t}(z) = H^p_t(z);\ \forall z \in [0, \bar{z}_t],$$

and those for fixed resources, unskilled labor, and human capital are

$$x_t L_{t-1} = X^p_t = X,$$
$$L^p_t = (1 - n_t(\tau + e_t))L_t,$$
$$\int_0^{\bar{z}_t} H^p_t(z)\mathrm{d}z + N_t\omega(\bar{z}_{t-1}) = (1 - n_t(\tau + e_t))L_t h_t,$$

where $X$ represents the economy's total endowment of fixed resources.

Two laws of motion are noteworthy. First, the population size of the current generation is equal to the population size of the previous generation multiplied by the fertility rate:

$$L_t = n_{t-1} L_{t-1}.$$

From here on, for brevity, we use the term "population" to refer to $L_t$, the population size of the young adults in the current period. Second, the current technological frontier is equal to the previous technological frontier plus the varieties of new goods:

$$\bar{z}_t = \bar{z}_{t-1} + \Delta(\bar{z}_{t-1}, N_t).$$

An equilibrium growth path consists of the optimal decisions of all agents, a price system, and a path of state variables which together satisfy the market-clearing conditions and laws of motion, given the initial state. The formal definition is provided in Appendix B.1.

**Lemma 3.** *Under Assumptions 1 and 2, the following results hold for* $\forall t \geq 1$ *in equilibrium:*

i. *When* $\bar{z}_{t-1} = 0$, $h_t > 0$ *iff* $L_{t-1} > \mathcal{Z}(0)$; *when* $\bar{z}_{t-1} > 0$, $h_t > 0$ *for sure.*

ii. *When* $\bar{z}_{t-1} = 0$, $N_t > 0$ *iff* $L_{t-1} > \mathcal{Z}(0)$; *when* $\bar{z}_{t-1} > 0$, $N_t > 0$ *iff*

$$(1-\rho)\rho^{\frac{\rho}{1-\rho}} \frac{1+\beta}{1+\beta+\beta\gamma} \cdot \frac{L_t h_t}{\bar{z}_{t-1}} > \frac{\omega(\bar{z}_{t-1})}{\delta_0(\bar{z}_{t-1})}. \tag{10}$$

*Here* $\mathcal{Z}(0)$ *is a constant, which is formally defined in Lemma 4 below.*

Lemma 3 describes two potential corner solutions in equilibrium. First, parents decide whether to invest in their child's human capital ($h_t = \eta(e_{t-1})$). Lemma 3.i states that, in the absence of modern technology, parents invest in children's human capital if and only if the population exceeds a threshold $\mathcal{Z}(0)$; in the presence of modern technology, parents always invest. Second, prospective developers choose whether to engage in R&D ($N_t = \int_0^\infty m_{it} \mathrm{d}i$). Lemma 3.ii states that without pre-developed modern technology, R&D also requires that the population exceeds the threshold $\mathcal{Z}(0)$; with pre-developed modern technology, R&D occurs if and only if the average human capital supply per pre-developed good ($L_t h_t / \bar{z}_{t-1}$) surpasses a threshold determined by the expected human capital inputs for a successful R&D ($\omega/\delta_0$).

Lemma 3 has two significant implications. First, there exists a Malthusian phase in equilibrium, characterized by no modern technology ($\bar{z}_{t-1} = 0$), no technological progress ($N_t = 0$), and no parental investment in human capital ($e_{t-1} = h_t = 0$). In this phase, parents focus solely on childbirth, leading to a high fertility rate and continuous population growth. The Malthusian phase persists until the population reaches a threshold where both R&D and human capital investment become beneficial. At this point, a modern growth starts. This result is consistent with Goodfriend and McDermott (1995) and Galor and Weil (2000).

Second, technological stagnation may arise even after leaving the Malthusian phase. Eq.(10) suggests that increasing competition for human capital between the production and R&D sectors can lead to stagnation. Since households prefer a diversity of goods, any increase in variety boosts the overall demand for modern goods. As the technological frontier advances, this demand growth pulls more human capital into the production sector, raising R&D costs. If the cost of human capital becomes excessively high compared with the potential profit from developing new goods, R&D halts and technological stagnation happens.

**Lemma 4.** *Under Assumptions 1 and 2, there exists a continuous function* $\mathcal{Z} : [0, \infty) \to (0, \infty)$ *such that in equilibrium,* $N_t > 0$ *iff* $L_{t-1} > \mathcal{Z}(\bar{z}_{t-1})$.

Lemma 4 refines Lemma 3.ii by showing that technological stagnation depends solely on the previous population $L_{t-1}$ and technological frontier $\bar{z}_{t-1}$. This result introduces a threshold population size for each level of technological frontier $\mathcal{Z}(\bar{z}_{t-1})$, above which R&D occurs. For example, $\mathcal{Z}(0)$ described in Lemma 3 is the threshold population size for R&D to occur in the absence of modern technology $\bar{z} = 0$. Lemma 4 implies that, within this framework, population $(L_t)$ is a more fundamental factor for technological progress than per capita human capital $h_t$. In Section 4, we provide a detailed analysis of the relationship between population and technological progress and further elaborate on this implication.

**Lemma 5.** *Under Assumptions 1 and 2, the current relative return of human capital* $s_t = r_t/w_t$ *can be uniquely determined by previous population* $L_{t-1}$ *and technological frontier* $\bar{z}_{t-1}$*:*

$$\frac{r_t}{w_t} = s(L_{t-1}, \bar{z}_{t-1}), \tag{11}$$

*where* $s$ *is a continuous function. Furthermore,*

i. *If* $\bar{z} = 0$ *and* $L \leq \mathcal{Z}(0)$*,* $s(L, \bar{z}) = [\tau\eta'(0)]^{-1}$*; otherwise* $s(L, \bar{z}) > [\tau\eta'(0)]^{-1}$.

ii. *If* $\bar{z} = 0$ *and* $L \leq \mathcal{Z}(0)$*,* $s_1(L, \bar{z}) = 0$*; otherwise* $s_1(L, \bar{z}) > 0$.

iii. *If* $L > \mathcal{Z}(\bar{z})$*, the sign of* $s_2(L, \bar{z})$ *is undetermined; otherwise* $s_2(L, \bar{z}) > 0$.

Lemma 5 pins down the relative return of human capital. Lemma 5.i states that, throughout the Malthusian phase, the relative return of human capital remains at $[\tau\eta'(0)]^{-1}$. Once the economy leaves the Malthusian phase, the relative return increases above $[\tau\eta'(0)]^{-1}$ to encourage positive parental time investment in children's human capital.

Lemma 5.ii states that after leaving the Malthusian phase, the relative return of human capital increases with the previous population. A larger population affects $s = r/w$ by, first, lowering the marginal product of unskilled labor, thus decreasing $w$ and increasing $s$; second, raising demand for modern goods, which boosts the demand for human capital, thus increasing $r$ and $s$; and third, expanding the supply of human capital, thus decreasing $r$ and $s$. While the third channel tends to lower $s$, it is outweighed by the first two. Due to the strong substitutability of households' demand across different goods ($\rho \in (0, 1)$),[12] changes in the human capital supply result in only small variations in the price of modern goods. This leads to a modest response of the equilibrium return of human capital to its supply, which renders the third channel dominated. Lemma 5.ii implies that population dynamics is self-regulated. A larger population raises the relative return of human capital, which renders investments in children's human capital more attractive and fertility less appealing. Changes in population size therefore move in the opposite direction of the level of population.

Lemma 5.iii implies that the relationship between the relative return of human capital and the technological frontier depends on whether R&D is occurring. With technology progressing, the demand for human capital arises from both the production and R&D sectors. While a more advanced technological frontier raises demand in the production sector, its effect on demand in the R&D sector is ambiguous, since we do not make any assumption on the relationship between the previous technological frontier and R&D ($\partial\Delta/\partial\bar{z}_{t-1}$, Assumption 2). Consequently, $\partial s_t/\partial\bar{z}_{t-1}$ is undetermined. In contrast, with technology stagnating, the demand for human capital is solely determined by the production sector, which leads to a positive relationship between the previous technological frontier and the relative return of human capital, i.e., $\partial s_t/\partial\bar{z}_{t-1} > 0$.

**Proposition 1.** *Under Assumptions 1 and 2, given any initial state* $\{L_0, \bar{z}_0, X\}$ *such that* $L_0 > 0$, $X > 0$, $\bar{z}_0 \geq 0$, *there exists a unique equilibrium growth path. Specifically, the economy is entirely described by two state variables, the population* $L_t$ *and the technological frontier* $\bar{z}_t$. *The equilibrium growth paths of the state variables are, for* $\forall t \geq 1$

$$L_t = \frac{\beta\gamma}{1 + \beta + \beta\gamma} \cdot \frac{L_{t-1}}{\tau + \varepsilon(r_t/w_t)}, \tag{12}$$

[12] $\rho \in (0, 1)$ means that the elasticity of substitution between households' consumption of different goods ($1/(1-\rho)$) lies in $(1, \infty)$. This is a standard and necessary assumption for solving the monopolist's problem faced by new technology developers (e.g., Romer, 1990).

$$\bar{z}_t = \Delta\left(\bar{z}_{t-1}, \Phi\left(C \cdot \left(\left(\frac{L_{t-1}}{\tau + \varepsilon(r_t/w_t)}\right)^{1-\alpha\rho} \cdot \left(\frac{r_t}{w_t}\right)^{-1}\right)^{\frac{1}{1-\rho}}, \bar{z}_{t-1}\right)\right) + \bar{z}_{t-1}, \tag{13}$$

*where $C$ is a positive constant and $r_t/w_t$ is determined by Eq.(11); the path $\{L_t, \bar{z}_t\}_{t=0}^{\infty}$ uniquely determines all other decisions and prices in equilibrium.*

Proposition 1 proves the existence and uniqueness of the equilibrium. Eq.(12) arises from the law of motion for the population, where $\beta\gamma/(1+\beta+\beta\gamma)$ represents the total time a parent allocates to household activities, and $\tau + \varepsilon(r_t/w_t)$ indicates the time spent on each child. Thus, $[\beta\gamma/(1+\beta+\beta\gamma)]/(\tau + \varepsilon(r_t/w_t))$ determines the fertility rate, $n_t$. Eq.(13) arises from the law of motion for the technological frontier, where $L_{t-1}/(\tau + \varepsilon(s_t))$ is proportional to current population $L_t$ according to Eq.(12), and the current population $L_t$ together with the prices $r_t/w_t$ determines the relative profit of a new good $\pi_t/r_t$. According to Lemma 2, $\Phi(\pi_t/r_t, \bar{z}_{t-1}) = N_t$ represents the measure of developers who participate in the R&D and $\Delta(\bar{z}_{t-1}, N_t)$ represents the variety of new goods developed.

# 4 The Dynamical System

This section characterizes, analyzes, and discusses the dynamics of the economy. We start by defining the population balance locus $\mathcal{LL}$ and the technology threshold locus $\mathcal{ZZ}$.

**Definition 1.** *The population balance locus $\mathcal{LL}$ is the set of state variables $(\bar{z}, L)$ for which the fertility rate is at the replacement level ($n = 1$) in equilibrium:*

$$\mathcal{LL} = \{(\bar{z}, L) : \tau + \varepsilon(s(L, \bar{z})) = \beta\gamma/(1+\beta+\beta\gamma)\}.$$

**Definition 2.** *The technological threshold locus $\mathcal{ZZ}$ is the set of state variables $(\bar{z}, L)$ for which the population is at the threshold for technological progress:*

$$\mathcal{ZZ} = \{(\bar{z}, L) : L = \mathcal{Z}(\bar{z})\},$$

*where $\mathcal{Z}(\cdot)$ is defined in Lemma 4.*

An additional assumption is introduced to ensure that $\mathcal{LL}$ is not empty. If the degree of parental altruism toward children is not sufficiently high and the costs of rearing child are excessive, the fertility rate could fall below replacement even without investing in children's human capital. Conversely, if parental altruism is overwhelming and the costs of rearing child are low, the fertility rate may remain above replacement even when the relative return to human capital is high. The following assumption rules out both extreme cases.

**Assumption 3.** *Suppose:* i. $\beta\gamma/(1+\beta+\beta\gamma) > \tau$. ii. $\beta\gamma/(1+\beta+\beta\gamma) < \tau+\bar{e}$, if there exists $\bar{e} > 0$ s.t. $\eta'(\bar{e})(\tau+\bar{e}) - \eta(\bar{e}) = 0$.[13]

**Proposition 2.** *Under Assumptions 1-3, the following results hold:*

i. *There exists a continuous function* $\mathcal{L} : [0,\infty) \to (0,\infty)$ *such that* $\mathcal{LL} = \{(\bar{z}_t, \mathcal{L}(\bar{z}_t))\}$.

ii. $\mathcal{LL} \bigcap \mathcal{ZZ} = \{(\dot{z}, \dot{L})\}$.

iii. $\mathcal{Z}'(z) > 0$, *for* $\forall z > 0$;
$\mathcal{L}'(z) < 0$, *for* $\forall z > \dot{z}$; $\text{sgn}(\mathcal{L}'(z)) = -\text{sgn}(s_2(\mathcal{L}(z), z))$, *for* $\forall z < \dot{z}$.

iv. $L_{t+1} > L_t$, $\bar{z}_{t+1} = \bar{z}_t$ *if* $L_t \in (0, \min\{\mathcal{Z}(\bar{z}_t), \mathcal{L}(\bar{z}_t)\})$;
$L_{t+1} > L_t$, $\bar{z}_{t+1} > \bar{z}_t$ *if* $L_t \in (\mathcal{Z}(\bar{z}_t), \mathcal{L}(\bar{z}_t))$;
$L_{t+1} < L_t$, $\bar{z}_{t+1} > \bar{z}_t$ *if* $L_t \in (\max\{\mathcal{Z}(\bar{z}_t), \mathcal{L}(\bar{z}_t)\}, \infty)$;
$L_{t+1} < L_t$, $\bar{z}_{t+1} = \bar{z}_t$ *if* $L_t \in (\mathcal{L}(\bar{z}_t), \mathcal{Z}(\bar{z}_t))$; *for* $\forall \bar{z}_t \geq 0$.

v. $(\bar{z}, L)$ *is a steady state iff* $\bar{z} \geq \dot{z}$, $L = \mathcal{L}(\bar{z})$.

Proposition 2 characterizes the dynamics of the economy. Proposition 2.i states that $\mathcal{LL}$ can be characterized by a threshold for population balance $\mathcal{L}(\bar{z})$. Proposition 2.ii states that $\mathcal{ZZ}$ and $\mathcal{LL}$ intersect at a unique point denoted $(\dot{z}, \dot{L})$. Proposition 2.iii states that $\mathcal{ZZ}$ is upward-sloping globally and $\mathcal{LL}$ is downward-sloping to the right of $(\dot{z}, \dot{L})$. To the left, $\mathcal{LL}$'s slope is undetermined and depends on the sign of $\partial s_{t+1}/\bar{z}_t$, which is not assumed in our paper (see Lemma 5). Fig.3 plots the simulated $\mathcal{LL}$, $\mathcal{ZZ}$, and growth paths. Appendix C.1 provides details on the simulation.

Given the current technological frontier $\bar{z}_t$, a state below $\mathcal{LL}$ ($L_t < \mathcal{L}(\bar{z}_t)$) indicates that the population is smaller than the threshold size for population shrinkage; the relative return of human capital is low, which encourages a higher fertility rate and population growth. Conversely, for states above $\mathcal{LL}$, the population contracts. A state above $\mathcal{ZZ}$ ($L_t > \mathcal{Z}(\bar{z}_t)$) indicates that the population is larger than the threshold size for technological progress, which encourages R&D. Conversely, for states below $\mathcal{ZZ}$, the technology stagnates.

[13] Whether $\bar{e}$ exists depends on the functional forms of $\eta(\cdot)$ and does not affect our results.

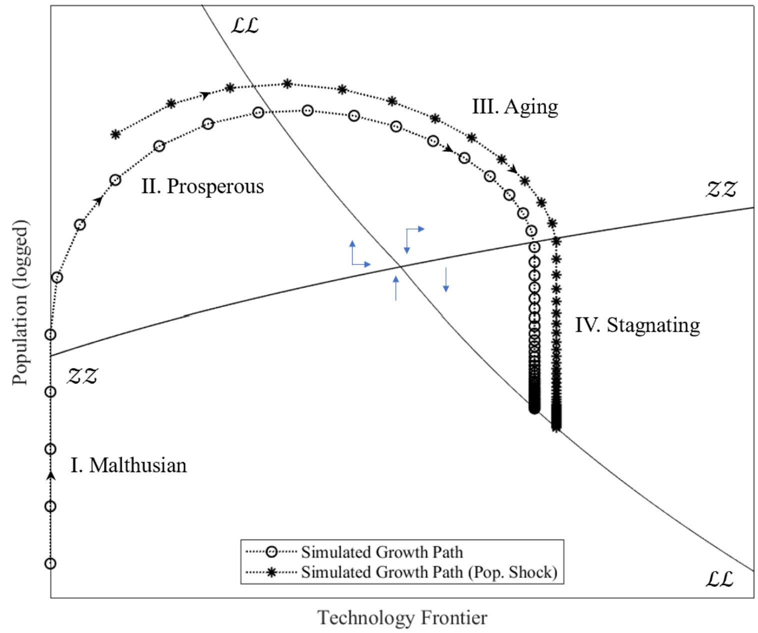



Figure 3: Phase Diagram

Consequently, Proposition 2.iv outlines four phases of the dynamic system—Malthusian, prosperous, aging, and stagnating—as shown in Fig.3, where the dotted line with circles plots the growth path of a typical economy. The economy begins in the Malthusian phase, where the state lies below both $\mathcal{LL}$ and $\mathcal{ZZ}$ ($L \in (0, \min\{\mathcal{Z}(\bar{z}), \mathcal{L}(\bar{z})\})$) with no modern technology ($\bar{z} = 0$). In this phase, there exists no technological advancement or human capital investment. Instead, the fertility rate is high, leading to continuous population growth.[14]

Once the population exceeds the threshold for technological progress, the economy shifts to the prosperous phase, in which the state lies below $\mathcal{LL}$ but above $\mathcal{ZZ}$ ($L_t \in (\mathcal{Z}(\bar{z}_t), \mathcal{L}(\bar{z}_t))$). During this phase, both technology and population grow, with human capital rising alongside increasing relative returns. The fertility rate begins above the replacement level but gradually declines as the economy develops, reflecting the process of demographic transition.

As the fertility rate drops below replacement, the economy shifts to the aging phase, where the state is above both $\mathcal{LL}$ and $\mathcal{ZZ}$ ($L_t \in (\max\{\mathcal{Z}(\bar{z}_t), \mathcal{L}(\bar{z}_t)\}, \infty)$). In this phase, technological progress continues, but the population starts to shrink. The age structure becomes skewed, with the number of older adults exceeding that of young adults in each

[14]If an exogenous technology shock happens to a Malthusian economy, it shifts to a state below both $\mathcal{LL}$ and $\mathcal{ZZ}$ but possesses some modern technology (to the right of the y-axis in Fig.3). We call these states quasi-Malthusian. A quasi-Malthusian economy moves in similarly to a Malthusian one, with expanding population and no technological progress.

period, which leads to the typical characteristics of an aging society. The pace of technological advances and growth in per capita income slows, because the aging population fails to provide an attractive market size and a sufficient total human capital supply.

When the population falls below the threshold for technological progress, R&D halts, and the economy enters the stagnating phase, with the state above $\mathcal{LL}$ but below $\mathcal{ZZ}$ ($L_t \in (\mathcal{L}(\bar{z}_t), \mathcal{Z}(\bar{z}_t))$). In this phase, technological stagnation sets in while the population continues to decline. As the relative return of human capital decreases due to the increasing wage of unskilled labor, the fertility rate slowly rebounds. Once it returns to the replacement level, the economy reaches a steady state.

Proposition 2.v shows that any state along the right arm of $\mathcal{LL}$, where both the population and the technological frontier remain constant, represents a steady state.

## 4.1 Analysis

***Population and technological progress.*** There are three channels through which population promotes R&D. First, a larger population increases total demand and market size, augments monopoly profits, and incentivizes R&D in the short run. The monopoly profit in equilibrium follows

$$\pi_t \propto r_t^{-\frac{\rho}{1-\rho}} \left( L_t c_{T,t}^y + L_{t-1} c_{T,t}^o \right) \propto r_t^{-\frac{\rho}{1-\rho}} L_t^{\alpha}.$$

This result demonstrates that $\pi_t$ increases with $L_t$. This prediction is consistent with empirical evidence in the literature (Acemoglu and Linn, 2004; Aghion et al., 2024).

Second, a larger population augments the total supply of human capital, reduces R&D costs, and also incentivizes R&D in the short run. Competition for human capital between the production and R&D sectors plays a vital role in determining R&D. If demand for human capital is purely driven by the production of pre-developed modern goods, the return of human capital follows:

$$r_t^{\frac{1}{1-\rho}} \propto \bar{z}_{t-1}/(L_t^{1-\alpha} h_t).$$

This result demonstrates that given the current per capita level of human capital $h_t$, a larger population relieves competition from the production sector and reduces R&D cost $r_t$. This channel becomes particularly significant when per capita human capital $h_t$ is high, given the concavity of the human capital production function $\eta(\cdot)$.

Third, a larger current population leads to a larger future population by the law of motion, thereby benefiting technological progress in both the short and long run.

To illustrate the long-run relationship between population and technological progress, Fig.3 also plots the simulated growth path after a one-time positive shock in population during the prosperous phase.[15] We make four observations. (1) The economy with the positive population shock (marked with stars) has a more advanced technological frontier in each period compared with the one without the shock (marked with circles). (2) Before stagnation occurs, the economy with the shock maintains a larger population at each level of technological development. (3) Once stagnation sets in, the economy with the shock experiences a faster decline in population. (4) Ultimately, the economy with the shock reaches a steady state characterized by more advanced technology, lower population, and higher per capita income.

***Inevitable permanent stagnation.*** Phase diagram analysis shows that an economy unavoidably falls into technological stagnation in the long run. This arises from three core ideas that are well established in the literature but haven't been jointly considered yet. First, permanent technological advancement requires sustained population growth (Jones, 1995a; Sasaki and Hoshida, 2017; Jones, 2022). By Eq.(10), continuous technological progress requires an ongoing increase in the total supply of human capital ($h_t L_t$). Since the production of per capita human capital exhibits diminishing returns ($\eta''(\cdot) < 0$), sustained population growth is necessary to maintain a rise in total human capital. Second, parental decisions regarding the trade-off between fertility (quantity) and child human capital (quality) are driven by the relative return of human capital to the unskilled wage (Becker et al., 1990; Galor and Mountford, 2008). Therefore, a fertility rate above replacement can persist only if the relative return remains below a certain threshold. Third, the relative return of human capital rises alongside enduring technological progress (Goldin and Katz, 1998; Autor et al., 1998). When technology reaches a sufficiently high level, the relative return of human capital inevitably becomes large enough to suppress the fertility rate below replacement, leading to population decline. This, in turn, eventually drives the economy into technological stagnation.

[15] A population shock during the Malthusian phase affects only the timing of the transition to modern growth, with little effect on the subsequent growth path. Fig.A.1 in Appendix C.2 depicts the impact of population shocks during the Malthusian phase.

***A market failure.*** Intuitively, the inevitability of stagnation stems from a market failure in pricing fertility. In the market, the relative price $r/w$ in the equilibrium reflects the scarcity of human capital relative to unskilled labor in the production and R&D sectors. This price is used by the representative and individual parent to make the optimal decision regarding the trade-off between fertility and the investment of human capital. This means that the individual parent is unable to internalize the broad positive externalities of fertility at the societal level, which includes expanding the size of the market, raising the supply of total human capital, and increasing the future population in our model. Consequently, with technological progress, the population declines, and the economy stagnates.

A key friction commonly emphasized in overlapping generations models is that households do not fully account for the welfare of future generations (Diamond, 1965). The market failure in pricing fertility within our model captures this friction, as households fail to internalize the impact of their fertility decisions on future population size. Our model also accommodates the inefficiency commonly highlighted in endogenous R&D models, where technological developers enjoy monopoly power. However, in our model, such inefficiency is less consequential compared to the market failure driven by the child quantity–quality trade-off—a point we will now discuss.

# 5 Discussion: The Central Role of the Child Quantity-Quality Trade-Off

The child quantity-quality trade-off is central in our model, because it connects two fundamentally choices in the economy—fertility and R&D—through human capital. The demand for human capital, influenced by firms' R&D decisions, impacts parental fertility decisions. Conversely, the supply of human capital, determined by parental decisions, subsequently affects firms' R&D decisions. Population and R&D are then jointly determined in equilibrium. Through the child quantity-quality trade-off, the two micro-foundations of our model—one on the household side and the other on the firm side—are naturally connected. The two connected micro-foundations thus enable our model to derive rich dynamics on both population and technology.

To better understand the central role of the child quantity-quality trade-off, in this section, we demonstrate that the unique predictions of our model—population aging and economic stagnation—hinge on this trade-off instead of other model specifications. We first show that, consistent with the literature, permanent technological progress and economic growth occur when we shut down the child quantity-quality trade-off in our model. We then discuss potential paths to achieve sustainable economic growth while considering the central role of the child quantity-quality trade-off.

## 5.1 Exogenous Fertility

We consider an environment that is identical to Section 2, except that the fertility rate is fixed. Specifically, the government taxes the return of human capital to subsidize the wage of unskilled labor—or, vice versa, to fix the relative return of human capital at a constant level, $\bar{s}$. The government ensures that $\bar{s}$ is sufficiently high to incentivize households to invest positively in their children's human capital but not excessive so that the fertility rate is above replacement:

$$\bar{h} = \eta(\varepsilon(\bar{s})) > 0, \;\; \bar{n} = \frac{\beta\gamma}{1+\beta+\beta\gamma} \cdot \frac{1}{\tau + \varepsilon(\bar{s})} > 1.$$

Assumption 3 guarantees the existence of such an $\bar{s}$. We assume that the birth policy is self-financing, meaning that the total subsidies provided equal the total taxes collected. Appendix D provides a detailed discussion of the tax and subsidy rates required to achieve these targets.

In order to directly compare with prior literature, we use the following functional forms for $\Delta$ and $\omega$ in this section:

$$\Delta(N_t, \bar{z}_{t-1}) = [(N_t + a)^\kappa - a^\kappa]\,(\bar{z}_{t-1} + b)^\psi, \;\; \omega(\bar{z}_{t-1}) = \omega_0(\bar{z}_{t-1} + b)^\psi,$$

where $\kappa \in (0,1)$, $\psi < 1$, $a > 0$, and $b > 0$. It is straightforward to verify that Assumption 2 is satisfied using these functions. In addition, these functional forms are consistent with those used in the literature (Jones, 1995a).[16]

With exogenous fertility, an equilibrium growth path consists of the optimal decisions of all agents, a price system, a path of state variables, and a birth policy maintaining the

[16]Compared with the functional forms in Jones (1995a), constant terms $a$ and $b$ are introduced here. This modification allows for the possibility of stagnation.

fertility rate at $\bar{n}$ which together satisfy market clearing conditions, laws of motion, and fiscal balance, given the initial state. The formal definition is provided in Appendix B.2.

**Proposition 3.** *Under Assumptions 1-3, given any initial state* $\{L_0, \bar{z}_0, X\}$ *such that* $L_0 > 0$, $X > 0$, $\bar{z}_0 \geq 0$, *there exists a unique equilibrium growth path. Also, the following results hold in equilibrium with exogenous fertility:*

i. *Stagnation occurs* ($\bar{z}_t = \bar{z}_{t-1}$) *iff:*

$$\bar{z}_{t-1} \geq \rho^{\frac{\rho}{1-\rho}} a^{\frac{\kappa-1}{\kappa}} \frac{\kappa}{\omega_0} \cdot \frac{(1-\rho)(1+\beta)}{1+\beta+\beta\gamma} \bar{h} L_0 \bar{n}^t. \tag{14}$$

ii. *There exists an* $\bar{n}_M > 1$ *such that for* $\forall \bar{n} \in (1, \bar{n}_M)$, *the economy converges to a balanced growth path:*

$$\lim_{t\to\infty} \frac{\bar{z}_t}{\bar{z}_{t-1}} = \bar{n}^{\kappa/(1-\psi+\kappa\psi)}, \quad \lim_{t\to\infty} \frac{y_t}{y_{t-1}} = \bar{n}^{(1-\rho)(\kappa/(1-\psi+\kappa\psi)+\alpha-1)},$$

*where* $y_t$ *refers to the per capita income in period* $t$.

Proposition 3 highlights the central role of the child quantity-quality trade-off in driving our main results on technological and economic stagnation. Proposition 3.i states that while technological stagnation remains possible under exogenous fertility, it becomes a temporary rather than a permanent state, in contrast to the case of endogenous fertility. In both cases, stagnation occurs due to the competition for human capital between production and R&D sectors. However, with exogenous fertility, the continuously growing population ensures that the supply of human capital ($\bar{h} L_0 \bar{n}^t$ in Eq.(14)) expands over time. This eventually alleviates the shortage of human capital, which renders stagnation only a temporary state.

Proposition 3.ii states that when the exogenous fertility rate is not excessively high, the economy converges to a balanced growth path. On this path, sustained population growth drives permanent increases in both the technological frontier and per capita income, with the long-term growth rate directly determined by the exogenous fertility rate. This result mirrors the findings of Jones (1995a), where population growth anchors long-term technological progress.[17]

[17] Slightly different from Jones (1995a), in our model the balanced growth path emerges asymptotically rather than immediately, due to the two constant terms, $a$ and $b$, in the functions of $\Delta$ and $\omega$. An upper

## 5.2 How to Achieve Sustained Growth

Our baseline result highlights the inevitability of stagnation due to the interplay between endogenous R&D and the child quantity-quality trade-off. First, firms' choice of R&D affects the relative return of human capital, which in turn affects parental decisions on fertility and each child's human capital investment; second, parents weigh the child quantity-quality trade-off, investing in each child's human capital at the cost of fertility; third, parental decisions determine future population and human capital supply, and thus influence R&D decisions, technological progress, and economic growth. Consequently, our analysis suggests three ways to interrupt the interplay and achieve sustained technological advancement and economic growth.

First, to decouple parental fertility decisions from the price signal—the relative return of human capital to unskilled labor, sent by the market—the government could implement active pronatalist population policies, such as fertility subsidies, childcare support, and extended maternal leave, to sustain population growth. These policies essentially help parents internalize the positive externality of fertility at the societal level, similar to our exercise in Section 5.1. Policies that encourage migration may also sustain population growth and thus help the economy achieve permanent growth.

The second way is to directly eliminate the child quantity-quality trade-off by improving the child human capital production technology. In our model setting, technological advancement is represented by the expanded variety of modern goods, holding the human capital production function ($\eta(\cdot)$) constant. If technological progress simultaneously improves the efficiency of human capital production, the required investment for a given level of human capital decreases and the opportunity cost of fertility decreases correspondingly. In this scenario, fertility does not necessarily decrease with technological progress.

Finally, to decouple firms' decision on R&D from the total supply of human capital, the government could encourage the integration of automation, machine learning, and artificial intelligence into both production and R&D processes to substitute for human capital. By doing so, the competition for human capital is alleviated between the production and

bound on the exogenous fertility rate ensures convergence regardless of parameter values or initial conditions. If the exogenous fertility rate is excessively high, while an asymptotic balanced growth path still exists, the growth rate may oscillate around the balanced path in the long term.

R&D sectors. Technological progress may then be sustainable even when the population is decreasing.

## 6 Future Extensions

Our research opens up three directions for further exploration. The first involves enriching our framework by modeling more household choices, such as marriage, old-age support, and bequest (Doepke and Tertilt, 2016), particularly considering that in traditional economies, parents' fertility decisions are also influenced by the expectation of receiving support from their children in old age. The second is to allow capital accumulation. To maintain model tractability, we currently assume that firms employ only human capital to produce modern goods or conduct R&D, and thus abstracting capital accumulation from our model. The third direction is to explore various innovation methods and technological progress. In our current model, R&D is the sole method of technological advancement in a closed economy. In the case of an open economy, technology could progress through imitation. Also, we represent technological progress only by increases in modern goods variety. Contemporary technological advancement occurs through multiple domains, such as improving human capital production efficiency or using artificial intelligence to substitute for human capital in knowledge production. Extending our framework in these three directions would help us better characterize the long-term relationship between population dynamics, technological progress, and economic growth. We relegate these extensions to our future research agenda.

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