北京大学中国经济研究中心
China Center for Ecnomic Research

讨论稿系列
Working Paper Series

E2022024 2022-12-25

Industry-optimal Information in a Search Market

Ju Hu

Abstract

This paper studies the welfare effects of information in a sequential consumer search environment. Consumers search in the market and observe a noisy signal about the match value upon being matched with a firm, based on which they make their purchasing decisions. We construct the class of conditional unit-elastic demand signal distributions such that every equilibrium that can possibly arise under a feasible signal distribution can be achieved by a signal distribution within this class, based on which we investigate the industry-optimal information in this market. Contrary to the conventional wisdom that lower search cost promotes competition and reduces industry surplus, we find that the optimal industry surplus is strictly increasing as the search cost decreases. We also characterize the industry-optimal signal distribution for a large class of value distributions.

Keywords: consumer search, information design, welfare limit, industryoptimal design, full surplus extraction

JEL Classification: D83, L11, L15

Industry-optimal Information in a Search Market*

Ju Hu †

First version: October 23, 2021
This version: December 14, 2022

Abstract

This paper studies the welfare effects of information in a sequential consumer search environment. Consumers search in the market and observe a noisy signal about the match value upon being matched with a firm, based on which they make their purchasing decisions. We construct the class of conditional unit-elastic demand signal distributions such that every equilibrium that can possibly arise under a feasible signal distribution can be achieved by a signal distribution within this class, based on which we investigate the industry-optimal information in this market. Contrary to the conventional wisdom that lower search cost promotes competition and reduces industry surplus, we find that the optimal industry surplus is strictly increasing as the search cost decreases. We also characterize the industry-optimal signal distribution for a large class of value distributions.

KEYWORDS: consumer search, information design, welfare limit, industry-optimal design, full surplus extraction

JEL CLASSIFICATION: D83, L11, L15

---

*I am grateful to the editor, Marco Battaglini, and to two anonymous referees for their comments and suggestions, which significantly improved the paper. I am also grateful to Angus Chu, Yucheng Ding, Fei Li, Jianpei Li, George Mailath, Allen Vong, Cheng Wang, and Jidong Zhou, as well as seminar participants at Northwestern Polytechnical University, University of Macau, and Wuhan University. I acknowledge the financial support of the NSFC (Grant No. 72273007). All remaining errors are mine.

†China Center for Economic Research, National School of Development, Peking University, 5 Yiheyuan Road, Haidian District, Beijing 100871, China; Email: juhu@nsd.pku.edu.cn.

1 Introduction

This paper investigates the welfare effect of centralized information disclosure in a model of sequential consumer search. Firms in the market simultaneously charge prices for their products. Consumers, each of whom has a unit demand for the products in the market, initially have imperfect information about the prices and values of firms' products. By incurring a search cost, a consumer can sequentially sample firms' products. Upon being matched with a firm, a consumer learns the price and observes a signal about the match value, based on which he decides whether to purchase or continue searching. The goal of this paper is to understand the welfare limits of all possible information disclosure rules, with a particular emphasis on the industry-optimal information policy.

Such questions are relevant when the platforms of online marketplaces, such as eBay and Amazon, consider the design of their platforms. Understanding their users' needs and their own business goals, these platforms carefully design their websites to organize, structure, and label contents in order to disclose the relevant information. These aspects of design will shape the consumers' perception of the listed products and services, which in turn will determine the market interactions and performance, and ultimately the profitability of the platforms. Moreover, an important amount of earnings for such platforms comes from the money they collect from sellers in the form of a certain fraction of the sales revenue. In such circumstances, it would be natural for platforms to adopt an industry-favoring approach for the underlying information design problem.

To understand the limits of the welfare effects of information in these markets, we adopt the information design approach and impose no parametric restriction on the form of information disclosure policies except that information is independent across firms' products. Such an approach also reflects the fact that the platforms can flexibly design their websites in practice.

We first construct a relatively simple class of feasible signal distributions, each of which induces an equilibrium in the search market. More importantly, we show that this class of signal distributions is in fact rich enough so that every equilibrium that can possibly arise under an arbitrary feasible signal distribution can be achieved by a signal distribution in this constructed class.

Each such signal distribution is parameterized by three values: a signal cutoff, the conditional mean of signals below the cutoff, and the corresponding consumer surplus in the induced equilibrium. Above the signal cutoff, there is a continuum of signals indicating high match values. Below the signal cutoff, there may be a single atom or another continuum of signals indicating low match values. In the induced equilibrium, the consumers actively search in the market until they receive a signal above the cutoff, in which case they purchase from the currently matched firm. Those continuum of signals are specially distributed so that given the consumers' equilibrium search behavior, each firm faces a demand curve with unit-elasticity over a certain range of prices. Such a signal distribution is referred to as a *conditional unit-elastic demand signal distribution*.

We then apply this construction to investigate the industry-optimal information. Contrary to the conventional wisdom that lower search cost reduces the industry surplus since it promotes competition among firms, we find that the industry surplus under the industry-optimal information is in fact strictly increasing as the market becomes less frictional. For fixed information disclosure, lower search cost leads to more search activity by consumers, which in turn intensifies competition among firms. However, when information disclosure can be flexibly adjusted according to the search cost, lower search cost does not necessarily imply more search activity. This is because consumers' search incentive also depends on the information disclosure rule. In particular, a less informative disclosure rule reduces search activity because products become more homogeneous across firms. Therefore, if a signal distribution induces a certain level of industry surplus in a market with high search cost, then in a market with low search cost, we find that a strictly higher industry surplus can be achieved by a less informative signal distribution.

Restricting attention to value distributions with increasing hazard rate, we show that the unique industry-optimal conditional unit-elastic demand signal distribution is the one that achieves the highest possible total welfare and gives all the surplus to the industry, provided the search cost is not too low. When the search cost is really low, such signal distribution is no longer feasible and there is typically a trade-off between total welfare and industry surplus. If the value distribution, in addition, has increasing density, the unique industry-optimal conditional unit-elastic demand signal distribution still allows the industry to extract all the equilibrium total welfare, but the total welfare is no longer maximized. This is because achieving the highest total welfare requires the consumers search intensively when the search cost is sufficiently low, which leads to a very competitive market. A lower total welfare can reduce consumers' search activity, thereby relaxing competition.

Related Literature This paper is closely related to Roesler and Szentes (2017), who study the buyer-optimal information in a monopoly pricing setting in the framework of Bayesian persuasion and information design (Rayo and Segal (2010), Kamenica and Gentzkow (2011)). They show that the maximal buyer surplus can be achieved by a signal distribution that induces a demand curve for the monopolist with unit-elasticity.[1] This paper extends their analysis to the current competitive and dynamic environment. Our construction of signal distributions generalizes their analysis by incorporating the consumers' endogenous outside option and search incentives. Moreover, in their setting, it is easy to see that disclosing no information at all is industry-optimal, in which case the firm simply charges the expected value and extracts all the surplus. In the search market, however, this is no longer true because no information disclosure will simply lead to an inactive market. Therefore, the industry-optimal information in our setting and its welfare consequence are much less obvious. We show that, under certain conditions, the industry-optimal information in our setting also results in maximized total welfare and full extraction by the firms, but it is achieved by carefully designed information.

Dogan and Hu (2022) study an information design problem in the same search framework as the current paper, focusing on the consumer-optimal design. They also construct a class of signal distributions that have the property of unit-elasticity and show that every feasible equilibrium *consumer surplus* can be achieved by a signal distribution in that class. But their construction suffers from the limitation that those signal distributions are not rich enough to achieve every feasible *equilibrium*. Consequently, their construction can not be applied to investigate the welfare limit of the search market, nor the industry-optimal information. The current paper complements their analysis by constructing a strictly larger class of signal distributions and showing that this larger class is sufficiently rich so that every feasible equilibrium can be achieved. This construction then allows studying the industry-optimal information. Section 3.2 discusses more about the comparison of the constructions, and Section 4.3 discusses the comparison of the consumer-optimal and industry-optimal information.

Armstrong and Zhou (2022) also study an information design problem in a competitive environment. Their main focus is duopolistic competition in a discrete choice

[1] Condorelli and Szentes (2020) analyze the classical hold-up problem, where there is no fixed prior about the buyer's valuation, and Choi et al. (2019), building on Anderson and Renault (2006), analyze the buyer-optimal information in a search good environment. Both studies also find that the unit-elasticity plays an important role in maximizing the buyer's welfare.

model. By considering the case where the market is always covered, which allows them to focus on information disclosure about the relative valuation, they fully characterize the consumer and industry optimal information, as well as the welfare limit of their market. Among other results, Armstrong and Zhou (2022) show that the industry-optimal information usually leads to maximized total welfare, which is clearly higher than the one under the consumer-optimal information. Our result is similar to theirs when the search cost is not too low, but differs when the search cost is sufficiently low. The major reason that drives the difference is the fact that information in the current model can affect market competition through the channel of consumers' search behavior, which is not present in their model. When the search cost is very low, it is very important to lower consumers' search incentive for the industry. It is then optimal not to achieve the highest possible total welfare, since lower total welfare can reduce consumers' search activity.

There are also other related papers studying decentralized information disclosure by competitive firms. Bar-Isaac et al. (2012) consider a similar consumer search model where firms compete not only in prices, but also their product designs. By restricting attention to a parameterized family of signal distributions that are ordered by the demand rotation order in Johnson and Myatt (2006), Bar-Isaac et al. (2012) show that every firm provides the extremal level of information: either the minimum or the maximum. Board and Lu (2018) study a search setting in which products across firms are homogeneous and the firms compete in how much information to disclose about the common state. Au and Whitmeyer (2018) consider a related information design problem in a directed search setting, where firms in an oligopolistic market compete in attracting and persuading buyers through their information disclosure about their own products of heterogeneous qualities. Au and Kawai (2020) analyze competition among firms that disclose their own product information to persuade buyers, but they abstract away firms' pricing behavior. Hwang et al. (2019) consider an oligopoly model in which firms compete not only in prices, but also their advertising strategies about how much product information to provide. Our setting is quite different from these papers, as firms only compete in their prices and product information disclosure is designed by a third party, such as a platform.

The remainder of this paper is organized as follows. Section 2 sets up the model. Section 3 contains the construction of signal distributions that can achieve every equilibrium under an arbitrary feasible signal distribution. Section 4 analyzes the industry-optimal signal distribution. Section 5 extends the analysis to equilibria in mixed strategies. Section 6 concludes. Unless otherwise stated, all the proofs are deferred to the appendix. We also provide an analysis of the welfare limit and some supplemental results for Section 4 in the online appendix.

2 Model

2.1 Setup

The model follows that in Dogan and Hu (2022), which is based on a model of sequential consumer search due to Wolinsky (1986). There are a continuum of risk neutral firms and a continuum of risk neutral consumers. Each firm supplies a single product. The firms' costs of providing their products are normalized to zero. Each consumer wishes to purchase one unit of one product from the market. The value of a firm's product to a consumer is u, which is distributed according to a cumulative distribution function F over $[0,1]$. Let μ denote the expected value, i.e., $\mu \equiv \int_0^1 u \mathrm{d}F(u) \in (0,1)$.

The market interaction is as follows. Firms simultaneously choose a price for their own product. Consumers must gather price and value information through a sequential search process. By incurring a search cost $s \in (0,\mu)$, a consumer is randomly matched with a firm, upon which he discovers the price, say p, and receives a noisy signal, say q, about the match value. Based on the signal, the consumer forms expectation $\mathbb{E}[u|q]$ about the match value and then decides whether to purchase from this currently matched firm. If he purchases, he stops searching and leaves the market. In this case, his expected surplus is $\mathbb{E}[u|q]-p$ and the profits of the matched firm are p. If he does not purchase, he and the firm get zero from the current match. He then can decide whether to continue searching.[2]

The main concern of this paper is how the product information available to consumers affects the welfare of this market, with a particular interest in what kind of information disclosure can maximize industry surplus. For this, we can think of a third party who, before the market opens, can design and commit to a product information disclosure rule, which specifies how signal q is correlated with the true product value u of the currently matched firm.[3] For example, if q and u are perfectly correlated, it is full information disclosure; if q and u are independent, it is

[2]Because we focus on a stationary environment, whether consumers have free recall or not does not matter.

[3]Formally speaking, an information disclosure rule consists of a (measurable) signal space Q and a system of conditional distributions $\{\nu(\cdot|u)\}_{u\in[0,1]}$ such that (i) for each u, $\nu(\cdot|u)$ is a probability measure over Q, which specifies the conditional distribution of signals given the true value u, and (ii) for each measurable $A \subset Q$, $\nu(A|\cdot) : [0,1] \to [0,1]$ is measurable.

no information disclosure at all. As mentioned in the introduction, one interpretation of this search market is an online platform. This platform controls how much product information is available to consumers through its web design. Such design can include, for example, how many pictures of each product to display, whether to provide detailed product specifications or just a short summary, whether to provide a free trial for digital contents, and so forth. Because the revenues of the marketplace platforms, such as eBay and Amazon, mainly come from charging a commission on transactions, a natural objective of the design of these platforms is to maximize the industry surplus.

Throughout the paper, we restrict attention to the case where signal q that a consumer observes when matched with a firm only reveals information about the current match value and is independent of the match values of other firms. We also assume that the information disclosure rule is identical across all matches. As consumers are risk neutral and their purchasing decisions only depend on the conditional mean $\mathbb{E}[u|q]$, the firms' demand and their pricing decisions are entirely driven by the marginal distribution of the conditional mean. Therefore, to understand the welfare effect of information disclosure and the industry-optimal design problem, it is sufficient to focus on the marginal distribution of the conditional mean that each disclosure rule induces. It is well known that a distribution of the conditional mean G is induced by some disclosure rule if and only if it is a *mean-preserving contraction* of F: $\int_0^1 q\mathrm{d}G(q) = \int_0^1 q\mathrm{d}F(q)$ and $\int_0^x G(q)\mathrm{d}q \leq \int_0^x F(q)\mathrm{d}q$ for all $x \in [0,1]$.[4] We refer to such a distribution G as a *feasible signal distribution* and let $\mathcal{G}_F$ be the set of all feasible signal distributions. The true value distribution F itself is feasible, representing the most informative signal distribution in $\mathcal{G}_F$. On the other hand, F_0, which specifies an atom of size one at μ, is also feasible, representing the totally uninformative signal distribution.

2.2 Equilibrium

Throughout most part of this paper, we focus on symmetric pure strategy equilibria with active search in this market.[5] In Section 5, we will extend our analysis to equilibria in mixed strategies and show that restricting attention to pure strategy equilibria

[4]See, for instance, Blackwell (1951), Gentzkow and Kamenica (2016), Kolotilin (2018), and Dworczak and Martini (2019).

[5]As usual in sequential search models, for any signal distribution $G \in \mathcal{G}_F$, there is always a trivial equilibrium where all firms set very high prices, e.g., $p \geq 1$, and consumers do not participate in the market at all. We rule out these uninteresting equilibria.

is indeed without loss of generality for industry-optimal design. A symmetric pure strategy equilibrium with active search, or simply equilibrium, consists of a price for the firms and a stopping rule for consumers, which satisfy the following two properties: (i) the consumers' stopping rule is optimal given that all firms charge the same equilibrium price, and (ii) no firm has an incentive to charge a different price given the consumers' stopping rule and the belief that all other firms charge the equilibrium price.

Lemma 1 below provides a simple characterization of an equilibrium.[6] Its statement needs a notation. For any signal distribution $G \in \mathcal{G}_F$, let $c_G : [0,1] \to \mathbb{R}_+$ be the consumers' *incremental benefit function*, defined as

$$c_G(x) \equiv \int_{[x,1]} (q - x)\,\mathrm{d}G(q), \;\; \forall x \in [0,1]. \tag{1}$$

Given G, the value $c_G(x)$ captures each consumer's incremental gain from one more search with a match of expected quality x at hand.

Lemma 1. *Suppose the signal distribution is $G \in \mathcal{G}_F$. A symmetric pure strategy equilibrium with active search is characterized by a pair (b, v) that satisfies the following two conditions:*

$$c_G(b) = s, \tag{2}$$

and

$$-\,c_G'(b-)(b - v) \geq -c_G'(x-)(x - v), \;\; \forall x \in [0,1], \tag{3}$$

On the equilibrium paths, each firm charges price $p = b - v$, and consumers purchase at p if and only if they receive a signal greater than or equal to b. The associated consumer surplus, industry surplus and total welfare, respectively, are v, p, and b.

Conditions (2) and (3) are the standard equilibrium characterization in the consumer search literature. Condition (2) characterizes the consumers' optimal stopping. When all firms charge the same price, consumers face a stationary environment. As a result, the consumers' optimal stopping rule is a cutoff rule. In particular, consumers stop and purchase from a firm with price p' and signal q if and only if $q - p' \geq v$, or equivalently $q \geq p' + v$, where v is consumers' continuation payoff from tomorrow on that serves as consumers' outside option when deciding whether to buy from the currently matched firm. Because the environment is stationary, this v is also consumers' equilibrium surplus. Since all firms charge the same price $p = b - v$ on the

[6]Because the result is standard, its proof is omitted.

equilibrium path, the consumers' equilibrium signal cutoff is then just b. Condition (2) characterizes this equilibrium signal cutoff. It states that with signal b at hand, the consumers are indifferent between stopping and one more round of search.

Condition (3) is about firms' incentives. It requires that no firm has an incentive to deviate from the equilibrium price $p = b - v$. The left hand side of (3) is a firm's profits from the equilibrium price. The right hand side is its profits from deviating to price $p' = x - v$. Facing such a price, the consumers' signal cutoff is just x. Thus, the firm's associated demand and profits are $-c'_G(x-) = 1 - G(x-)$ and $-c'_G(x-)(x - v)$, respectively.[7] Therefore, firms' pricing incentive can be equivalently interpreted as no firm having an incentive to deviate to a different signal cutoff x than b.

Since consumers have unit demand and purchase for sure in equilibrium, the equilibrium price p is also equal to the equilibrium industry surplus. Therefore, the expected total welfare, which is the sum of consumer surplus and industry surplus, is then $v + p = b$.[8]

Not every signal distribution $G \in \mathcal{G}_F$ induces an equilibrium with active search. It is also possible that no signal distribution induces such an equilibrium. The search market (F, s) *admits active search* if at least one feasible signal distribution $G \in \mathcal{G}_F$ induces such an equilibrium. In such a market, different signal distributions lead to different equilibria and thus different levels of total welfare and its division between the consumers and the industry. The goal is to understand the set of all achievable equilibria and the industry-optimal signal distribution.

2.3 Feasible incremental benefit functions

The previous definition of equilibrium indicates that the effects of a signal distribution on the corresponding equilibrium behavior are completely summarized by its incremental benefit function. Therefore, to understand all achievable equilibria, it is useful to understand the set of all feasible incremental benefit functions. In this way, we can work directly with incremental benefit functions without referring to the underlying signal distribution. The following lemma from Dogan and Hu (2022), which builds on Gentzkow and Kamenica (2016), provides a convenient characterization, which allows a simple geometric representation.

[7]By integration by parts, $c_G(x) = \int_x^1 (1 - G(q))\mathrm{d}q$. Hence, $-c'_G(x-) = 1 - G(x-)$.

[8]Another way to see this is to notice that total welfare by definition equals $\mathbb{E}_G[q|q \geq b] - \frac{s}{1-G(b-)}$, where the first term is the expected match quality and the second term is the total search expenditures. Using equilibrium condition (2), it is easy to verify that this expression equals b, which is equal to the equilibrium signal cutoff.

Lemma 2. *A function $c : [0,1] \to \mathbb{R}$ is the incremental benefit function for some signal distribution $G \in \mathcal{G}_F$, i.e., $c \in \mathcal{C}_F$, if and only if it is convex and $c_{F_0} \leq c \leq c_F$.*

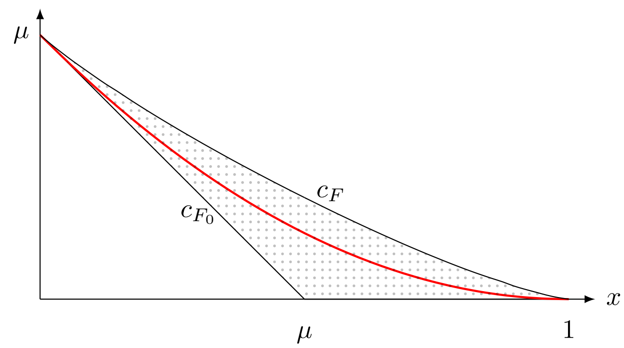


Figure 1: Feasible incremental benefit functions

Lemma 2 can be easily understood by looking at Figure 1. The lower solid curve is c_{F_0}. It is a downward-sloping-45-degree line over $[0, \mu]$ and becomes zero over $[\mu, 1]$. The higher solid curve is c_F. The shaded area between c_{F_0} and c_F represents the range of $\mathcal{C}_F$. Any convex function in this area is in $\mathcal{C}_F$ and thus is an incremental benefit function for some feasible signal distribution, and vice versa. The red curve is an example of such a function. We point out that two geometric features of an incremental benefit function can directly tell us some properties of its underlying signal distribution, which will facilitate understanding our construction in the next section. First, a kink point on an incremental benefit function means an atom of its underlying signal distribution at this point.[9] The kink at μ of c_{F_0} is an example. Second, a straight line segment over a certain interval, say $[x, x']$, means that G has no mass over (x, x').[10] The straight line segments over $[0, \mu]$ and $[\mu, 1]$ of c_{F_0} are two examples.

3 Conditional unit-elastic demand signal distribution

In this section, we construct a special class of signal distributions in terms of their corresponding incremental benefit functions. Each such signal distribution will induce an equilibrium with active search. More importantly, this class of signal distributions

[9]This is because $G(x) - G(x-) = (1 + c'(x+)) - (1 + c'(x-)) = c'(x+) - c'(x-)$.

[10]This is because $G(x'-) = 1 + c'(x'-) = 1 + c'(x+) = G(x)$.

is rich enough so that every equilibrium that can arise under an arbitrary feasible signal distribution can be achieved by a signal distribution in this class. Therefore, we can focus on this class of signal distributions to analyze the industry-optimal signal distribution in Section 4.

3.1 Construction

Every incremental benefit function to be constructed is parameterized by three values: $b \in [\mu - s,\, \bar{b}]$, $v \in [0, b)$ and $a \in [0,\, \mu - s)$, where $\bar{b}$ is the solution to $c_F(\bar{b}) = s$. Such an incremental benefit function will induce equilibrium (b, v), and parameter a measures the conditional mean of the signals below b, i.e., $\mathbb{E}[q|q < b]$, under the corresponding signal distribution. Let $\rho \equiv \frac{\mu - s - a}{b - a}$, which will be the probability of trade per match. Let $\pi \equiv \rho(b - v) = \frac{\mu - s - a}{b - a}(b - v)$, which will be the firm's expected profits in equilibrium. The construction distinguishes two cases. If $a \leq v + \pi$, define

$$c_{a,b,v}(x) \equiv \begin{cases} \mu - x, & \text{if } x \in [0, a], \\ \ell(x), & \text{if } x \in (a, b], \\ \max\bigl\{\bar{h}(x),\, 0\bigr\}, & \text{if } x \in (b, 1], \end{cases} \tag{4}$$

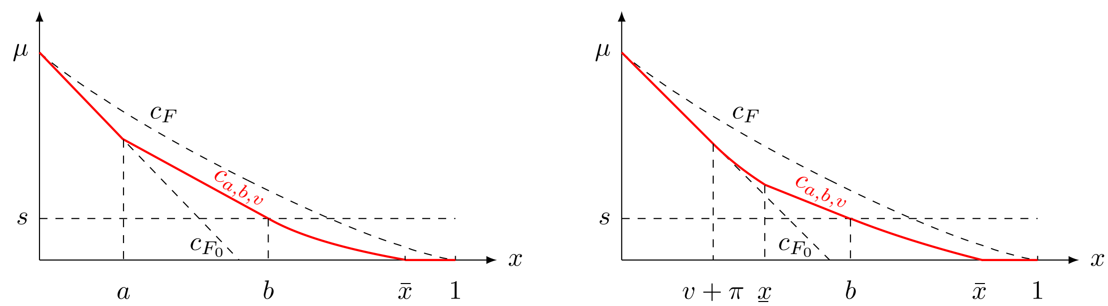
where $\ell(x) \equiv s - \rho(x - b)$ and $\bar{h}(x) \equiv s - \pi \log \frac{x - v}{b - v}$. If $a > v + \pi$, define

$$c_{a,b,v}(x) \equiv \begin{cases} \mu - x, & \text{if } x \in [0,\, v + \pi], \\ \max\{\underline{h}(x),\, \ell(x)\}, & \text{if } x \in (v + \pi,\, b], \\ \max\bigl\{\bar{h}(x),\, 0\bigr\}, & \text{if } x \in (b, 1], \end{cases} \tag{5}$$

where $\underline{h}(x) \equiv \mu - v - \pi - \pi \log \frac{x - v}{\pi}$.

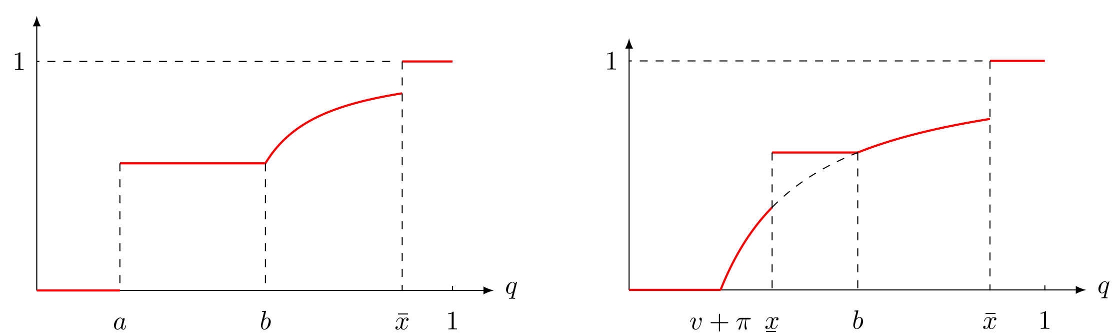
Figure 2 illustrates typical $c_{a,b,v}$'s for both cases $a \leq v + \pi$ and $a > v + \pi$, as well as their corresponding signal distributions. Panel (a) illustrates the case $a \leq v + \pi$. It coincides with the lower bound c_{F_0} over $[0, a]$, and is the straight line $\ell(x)$ over $[a, b]$. Over $[b, 1]$, it takes the particular functional form $\max\{\bar{h}(x), 0\}$. As will be seen soon, such construction will make the firms different over a certain range of prices in equilibrium. Panel (c) depicts the corresponding signal distribution. Below b, there is only an atom at a.[11] Above b, there is a continuum of signals. There is potentially an atom at some $\bar{x} \in (b, 1]$. It arises because of the truncation of $\bar{h}$, i.e., $\bar{h}(\bar{x}) = 0$. For ease of exposition, we refer to this type of $c_{a,b,v}$ as *type I*.

[11] When $b = \mu - s$, this atom disappears. In this case, the choice of a is irrelevant.



(a) $c_{a,b,v}$ if $a \leq v + \pi$

(b) $c_{a,b,v}$ if $a > v + \pi$



(c) Signal distribution if $a \leq v + \pi$

(d) Signal distribution if $a > v + \pi$

Figure 2: Construction of $c_{a,b,v}$ and the corresponding signal distribution

Panel (b) illustrates the constructed $c_{a,b,v}$ when $a > v + \pi$. It takes a similar form as the one of type I over the intervals $[0, v + \pi]$ and $[b, 1]$. The difference arises in the interval $[v+\pi, b]$. When $a > v+\pi$, $c_{a,b,v}$ is not a straight line over $[v+\pi, b]$. Rather, it takes the form of $\max\{\underline{h}(x), \ell(x)\}$. Such construction is also to guarantee that firms are indifferent over a certain range of prices in equilibrium. Panel (d) depicts the corresponding signal distribution. Unlike the one in panel (c) where there is an atom at a, this atom is replaced by a continuum of signals over $[v + \pi, \underline{x}]$.[12] The potential atom $\underline{x}$ arises from the truncation of $\underline{h}$ by ℓ, i.e., $\underline{h}(\underline{x}) = \ell(\underline{x})$. We refer to this type of $c_{a,b,v}$ as *type II*.

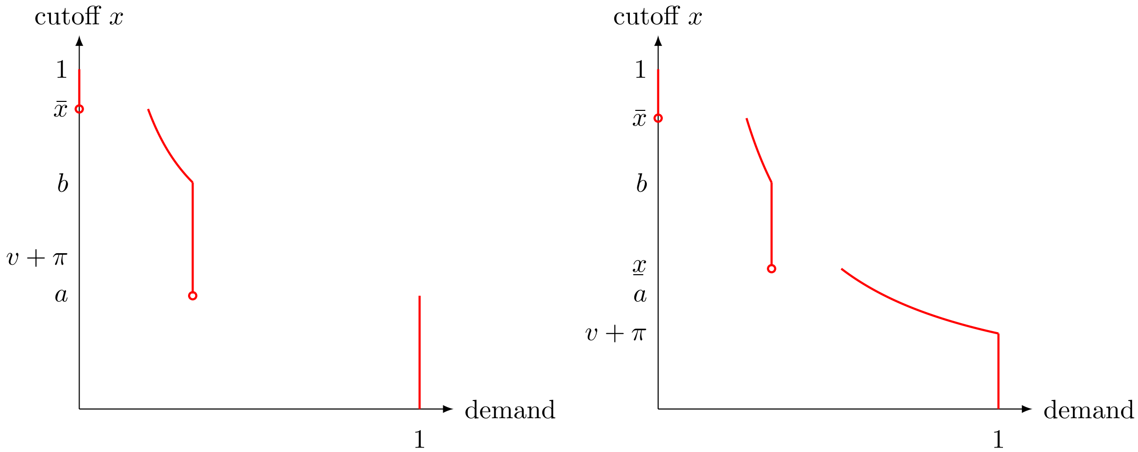
By construction, if $c_{a,b,v}$ is type I, it is above c_{F_0} and convex. Thus, it is feasible if it is below c_F. If $c_{a,b,v}$ is type II, it is still above c_{F_0} by construction. It is convex if and only if $\underline{h}(b) \leq s$, which rules out the possibility that $\underline{h}$ is always above ℓ over $[v+\pi,\ b]$. Hence, such $c_{a,b,v}$ is feasible if and only if $\underline{h}(b) \leq s$ and $c_{a,b,v} \leq c_F$. Let $\mathcal{U} \subset \mathcal{C}_F$ be the set of all feasible $c_{a,b,v}$'s. The following proposition explains the significance of the above construction.

Proposition 1. *Every $c_{a,b,v} \in \mathcal{U}$ induces equilibrium (b, v), in which the probability of trade per match is ρ and the firm's expected profits are π. Conversely, if (b, v) is an equilibrium under feasible signal distribution G, then $c_{a,b,v} \in \mathcal{U}$, where $a \equiv \mathbb{E}_G[q|q < b]$.*

The results of Proposition 1 are twofold. On the one hand, it verifies that the construction of $c_{a,b,v}$ is "correct" in the sense that it indeed induces equilibrium (b, v). On the other hand, it asserts that this constructed special class of signal distributions is in fact rich enough to induce every equilibrium that can possibly arise under an arbitrary signal distribution. These two parts together greatly simplify the analysis of the industry-optimal signal distribution, because they allow us to restrict attention to a relatively simple and parameterized class of signal distributions.

To gain some intuition about the first part of Proposition 1, observe first that $c_{a,b,v}(b) = s$ by construction. Hence, equilibrium condition (2) for the consumers' search incentive is satisfied. To understand why it is optimal for firms to set signal cutoff b, or equivalently charge price $b - v$, it is useful to examine the demand curve that a matched firm faces under $c_{a,b,v}$. Panel (a) in Figure 3 gives an illustration for a type I $c_{a,b,v}$, while panel (b) gives an illustration for a type II $c_{a,b,v}$. In both panels,

[12] Observe that ℓ is the left tangent line of $c_{a,b,v}$ at b. Because ℓ intersects the downward-sloping-45-degree line at a by construction, a is just the conditional mean of signals below b, as is the case of type I signal distributions.



(a) $c_{a,b,v}$ is type I

(b) $c_{a,b,v}$ is type II

Figure 3: Induced demand curve under $c_{a,b,v}$

the vertical axis represents the signal cutoff x. Given the consumers' equilibrium search behavior, a firm setting cutoff x is equivalent to charging price $x - v$. The horizontal axis represents the associated demand $-c'_{a,b,v}(x-)$, i.e., the probability that the consumer's signal realization is greater than or equal to x.

Consider a type I $c_{a,b,v}$ first. It is so constructed that its induced demand curve in panel (a) has unit elasticity when the signal cutoff changes over the range $[b, \bar{x}]$. Hence, every firm is in fact indifferent between any signal cutoff in this range. Moreover, because $a \leq v + \pi$, where π is the expected profits when setting signal cutoff b, the expected profit from signal cutoff a is $a - v \leq \pi$. Therefore, setting signal cutoff b, or equivalently charging price $b - v$, is optimal for firms. When $c_{a,b,v}$ is type II, it is so constructed that its induced demand curve in panel (b) has unit elasticity when the signal cutoff changes over the range $[v + \pi, \underline{x}] \cup [b, \bar{x}]$. It is then optimal for firms to set signal cutoff b given such a demand curve.

For the second part of Proposition 1, we show that if (b, v) is an equilibrium under feasible signal distribution G, then $c_{a,b,v} \leq c_G$, where $a = \mathbb{E}_G[q|q < b]$. That is, the signal distribution under $c_{a,b,v}$ must be less dispersed than G in the sense of mean preserving contraction, which in turn implies that $c_{a,b,v}$ is feasible. Essentially, the signal distribution under $c_{a,b,v}$ is obtained by modifying G in two ways. First, the signal distribution over $[b, 1]$ under G is replaced by a signal distribution that induces unit-elastic demand. Second, the signal distribution over $[0, b)$ under G is replaced by the atom signal $a = \mathbb{E}_G[q|q < b]$ if $c_{a,b,v}$ is type I, or a signal distribution that again induces unit-elastic demand if $c_{a,b,v}$ is type II. Both modifications make the signals more concentrated than G. Hence, the distribution under $c_{a,b,v}$ is a mean preserving spread of G.

A direct corollary of Proposition 1 is an intuitive characterization of when the search market (F, s) admits active search.

Lemma 3. *The search market (F, s) admits active search if and only if $s \in (0, \bar{s}]$ for some $\bar{s} \in (0, \mu)$.*

For any true value distribution F, there exists a *search cost threshold for active search* $\bar{s}$. When the search cost s is less than $\bar{s}$, by properly designing the information available to the consumers, there is always an equilibrium in which the consumers search actively in this market. But when the search cost s is greater than this threshold, the market is always inactive, regardless of what kind information is available to the consumers.

3.2 Discussion

The above construction of $c_{a,b,v}$ is built on Roesler and Szentes (2017), who first find that, in their monopoly pricing setting, modifying any given signal distribution into one that induces a unit-elastic demand curve does not change the monopolist's pricing incentive but requires less information disclosure. Based on this observation, they show that the consumer surplus in this market can be maximized by a unit-elastic demand signal distribution. Under this consumer-optimal signal distribution, the monopolist is indifferent between prices in the support of this distribution, and trade occurs with probability one in the consumer-optimal equilibrium. The previously constructed incremental benefit function $c_{a,b,v}$ also shares this feature. In equilibrium, firms are indifferent between prices over a certain range no matter whether it is type I or type II. But $c_{a,b,v}$ generalizes their unit-elastic demand signal distribution in two ways that reflect the nature of the current dynamic and competitive search environment. First, by different combinations of a and b, the constructed $c_{a,b,v}$ allows for different probabilities of trade per match. This is a necessary component in the current setting, as the consumers always have the opportunity to find a better match. Second, consumers' outside option, which coincides with the consumer surplus v and determines their search behavior and the competition between firms, is endogenous. Thus, it is incorporated as part of the design.

The currently constructed conditional unit-elastic demand signal distributions are a strictly broader class of signal distributions than those constructed in Dogan and

Hu (2022). Dogan and Hu (2022) only construct $c_{a,b,v}$'s of type I with the additional restriction that $a \leq v$. Although these signal distributions serve their purpose in that they can achieve every *consumer surplus* that can arise under an arbitrary feasible signal distribution, they are not rich enough to achieve *every equilibrium*. The reason is straightforward. The restriction $a \leq v$ imposes an exogenous upper bound $b - a$ on the equilibrium price that every $c_{a,b,v}$ can induce. For instance, for some total welfare level b, such restriction may rule out the possibility of full surplus extraction by the industry, because the parameter a may not be zero given b due to the feasibility constraint.[13] As a result, the class of signal distributions constructed in Dogan and Hu (2022) can not be used in studying the industry-optimal information design. The current construction solves this limitation by introducing $c_{a,b,v}$'s type II. As will be seen soon, the industry-optimal signal distribution is indeed of type II in many cases.

Following the terminology in Dogan and Hu (2022), the underlying signal distribution that gives rise to such an incremental benefit function $c_{a,b,v}$ is referred to as the *conditional unit-elastic demand signal distribution*.

4 Industry-optimal Design

By Proposition 1, the set of all achievable equilibria and thus the welfare limits of this search market can be analyzed by focusing on the class of conditional unit-elastic demand signal distributions, which is much smaller and simpler than the class of all feasible signal distributions. The online appendix provides the analysis. In this section, we focus on industry-optimal design.

4.1 Industry-optimal signal distribution

As we have explained in Section 2.2, the industry surplus in an equilibrium (b, v) equals the corresponding equilibrium price $b-v$, because trade occurs with probability one and consumers have unit demand. Therefore, finding a signal distribution that maximizes the industry surplus is equivalent to finding one that leads to the highest equilibrium price. For this goal, Proposition 1 implies that we only need to consider the following problem:

$$\max_{c_{a,b,v} \in \mathcal{U}} b - v \tag{6}$$

Recall that $\bar{b}$ is the unique solution to $c_F(\bar{b}) = s$. It is an upper bound of all

[13] See (7) in Section 4.2.

achievable total welfare in this market. Let $\hat{b} \in [\mu - s, \bar{b}]$ be the highest achievable total welfare in this market. Claim A.2 in the online appendix shows that the set of all achievable total welfare is interval $[\mu - s, \hat{b}]$. Therefore, problem (6) can be thought of as a two stage optimization problem by rewriting it as

$$\max_{b \in [\mu - s,\, \hat{b}]} \big(b - \min_{c_{a,b,v} \in \mathcal{U}} v\big).$$

For each achievable total welfare $b \in [\mu - s,\, \hat{b}]$, $\min_{c_{a,b,v} \in \mathcal{U}} v$ is the associated minimal feasible consumer surplus. Then, $b - \min_{c_{a,b,v} \in \mathcal{U}} v$ corresponds to the highest possible equilibrium price for total welfare b. Among all the feasible levels of total welfare, the industry-optimal signal distribution should choose the one that maximizes $b - \min_{c_{a,b,v} \in \mathcal{U}} v$. There is another intuitive interpretation of this problem. Recall that consumer surplus v also serves as consumers' outside option when matched with a firm. Hence, given b, the problem $\min_{c_{a,b,v} \in \mathcal{U}} v$ can be thought as finding the signal distribution that gives consumers the lowest outside option among those that lead to total welfare b. Because each firm is competing with consumers' outside option, finding the lowest outside option simply means finding the least competitive market.

Let $\tilde{b} \in [\mu - s,\, \hat{b}]$ be the highest possible total welfare at which full surplus extraction by the industry is feasible. That is, $\tilde{b}$ is the largest b such that $c_{a,b,0}$ is feasible for some a.[14] If $\tilde{b} = \hat{b}$, even at the highest achievable total welfare $\hat{b}$, full surplus extraction by the industry is feasible. Then, the industry-optimal surplus is achieved by $c_{a,\hat{b},0}$ for some a. If $\tilde{b} < \hat{b}$, it is clear that the total welfare under the industry-optimal signal distribution can only appear in $[\tilde{b}, \hat{b}]$, because for any $b < \tilde{b}$, the highest industry surplus is bounded above by b. Higher $b \in [\tilde{b}, \hat{b}]$ leads to higher total welfare, but at the same time, it may also lead to higher minimal feasible consumer surplus $\min_{c_{a,b,v} \in \mathcal{U}} v$. Therefore, the total welfare level that maximizes the industry surplus must resolve this trade-off.

For general value distribution F, it is very difficult to characterize $\hat{b}$, $\tilde{b}$, and the total welfare that maximizes the industry surplus in the case of $\tilde{b} < \hat{b}$. This is because the feasibility constraint involves uncountably many inequality constraints. Despite this difficulty, it is still possible to compare the optimal industry surplus, i.e., the value of problem (6), for different search costs. Quite surprisingly, the following proposition shows that the industry surplus under the optimal design is increasing as the search cost decreases.

[14] Because $c_{0,\mu-s,0}$ is always feasible when market (F, s) admits active search, such $\tilde{b}$ exists. See the proof of Lemma 3 in Appendix A.

Proposition 2. *For any value distribution F, the industry surplus under the optimal signal distribution is strictly increasing as the search cost decreases over* $(0, \bar{s}]$.

Contrary to the traditional wisdom that lower search cost leads to fiercer competition among firms, which would reduce the equilibrium price and thus industry surplus, Proposition 2 states that lower search cost is beneficial for the industry. As long as the information available to the consumers can be adjusted flexibly and properly as the search cost changes, the industry surplus is in fact strictly higher in the market with lower search cost.

The proof of Proposition 2 shows that if a certain level of industry surplus is achieved by some signal distribution in the market with search cost s, then for any $s' < s$, a strictly higher industry surplus can be achieved by a less informative signal distribution in the market with search cost s'. The underlying logic is the following. The standard intuition that lower search cost intensifies competition is based on the idea that lower search cost leads to more search activity by consumers. This is clearly true if the product information available to consumers does not change with search cost, for example, if it is always full information disclosure as in Wolinsky (1986) and Anderson and Renault (1999). However, as Anderson and Renault (1999) point out, how much information is available to consumers is also a determinant of their search behavior. For instance, less information means more homogeneity among firms, which in turn lowers consumers' search incentive. Therefore, if information disclosure can be adjusted flexibly as the search cost changes, then disclosing less information in a market with lower search cost can be used to offset consumers' additional search incentive due to lower search cost, thereby softening competition among firms. There may be concern that less information can potentially intensify rather than soften competition, because products become more homogeneous. But Proposition 2 shows that if "less information" is properly designed, the overall effect on softening competition is positive, leading to a strictly higher equilibrium price and thus higher industry surplus in the market with lower search cost.

4.2 Value distributions with increasing hazard rate

In this subsection, we restrict attention to value distributions with increasing hazard rate. This class of value distributions makes problem (6) more tractable, and the results will illustrate our analysis in Section 4.1. Formally, we assume that value function F has a positive and continuous density f over $(0, 1)$ such that its hazard rate function $\frac{f}{1-F}$ is increasing. These regularity conditions are widely used in the consumer search literature.[15] The increasing hazard rate property is equivalent to log-concavity of the survival function $1-F$. A sufficient condition is that f is increasing. Another well-known condition is that f itself is log-concave.[16]

Anderson and Renault (1999) show that when the search cost s is less than or equal to a cutoff $\hat{s}$, full information disclosure in this market leads to an equilibrium with active search. The corresponding signal cutoff, or equivalently total welfare, is just $\bar{b}$. The equilibrium price is $\frac{1-F(\bar{b})}{f(\bar{b})}$ and the consumer surplus is $\bar{b}-\frac{1-F(\bar{b})}{f(\bar{b})}$. Clearly, $\hat{s}$ is the search cost determined by $\bar{b}=\frac{1-F(\bar{b})}{f(\bar{b})}$.[17] Recall that $\bar{s}$ denotes the threshold for active search in Lemma 3. By definition, we have $\bar{s}\geq\hat{s}$. Claim B.1 in the online appendix shows that it must be $\bar{s}>\hat{s}$. Recall that $\hat{b}$ denotes the highest achievable total welfare that can possibly arise under an arbitrary feasible signal distribution.[18] When $s\in(0,\hat{s}]$, we clearly have $\hat{b}=\bar{b}$. When $s\in(\hat{s},\bar{s}]$, Claim B.2 in the online appendix shows that $\hat{b}<\bar{b}$.

Proposition 3 below finds the unique industry-optimal signal distribution as long as the search cost is not too small. Its statement requires introducing some notation. Recall that the equilibrium probability of trade per match under $c_{a,b,v}$ is $\rho=\frac{\mu-s-a}{b-a}$. For a given $b>\mu-s$, this probability is strictly decreasing in a. This is obvious from panel (a) of Figure 4. This probability of trade corresponds to the slope (in absolute value) of the line segment connecting $(a,\mu-a)$ and (b,s). For instance, because $a'<a''$ in this graph, the associated slope and thus probability of trade for a' is greater than that for a''. However, it is also obvious from this graph that there is no feasible $c_{a',b,v}$ for any v, because the solid red line segment, which constitutes part of $c_{a',b,v}$, is sometimes higher than c_F. Hence, for this b, there is a highest feasible probability of trade, and it is obtained by setting a to its lowest feasible level:

$$a(b)\equiv\min\left\{a\in[0,\,\mu-s)\,\middle|\,s-\frac{\mu-s-a}{b-a}(x-b)\leq c_F(x),\ \forall x\in[a,b]\right\}. \tag{7}$$

The set on the right hand side of (7) is the set of all feasible a's, as the inequalities simply require that the line segment connecting $(a,\mu-a)$ and (b,s) be below c_F. At the lowest feasible $a(b)$, some of these inequalities must be binding, as is illustrated by the blue line segment in panel (a) of Figure 4. Panel (b) illustrates the special

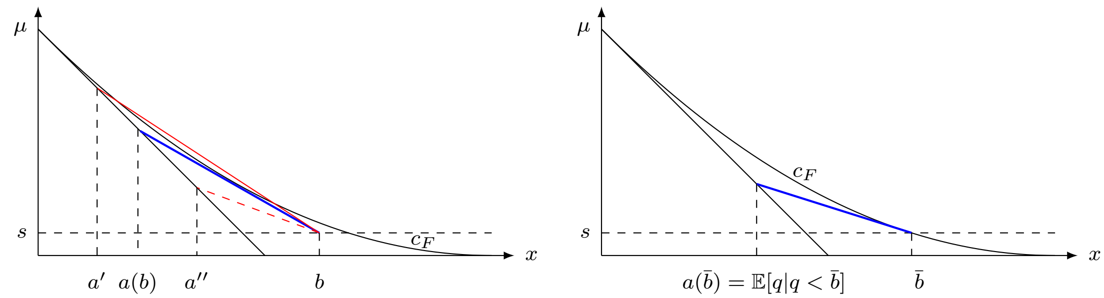
[15] See, for example, Anderson and Renault (1999), Armstrong et al. (2009), Eliaz and Spiegler (2011), Choi et al. (2018).

[16] See, for instance, An (1998) and Bagnoli and Bergstrom (2005).

[17] Because $c_F(\bar{b})=s$, $\bar{b}$ obviously depends on search cost s. For notational simplicity, we suppress s from $\bar{b}$.

[18] As $\bar{b}$, $\hat{b}$ also depends on search cost s. We also suppress s from $\hat{b}$ for notational simplicity.

case where $b = \bar{b}$. In this case, $a(\bar{b}) = \mathbb{E}[q|q < \bar{b}]$, i.e., the conditional mean of values below $\bar{b}$ under the value distribution F. The corresponding blue line segment is just the tangent line of c_F at $\bar{b}$, and the highest feasible probability of trade is then $-c_F'(\bar{b}) = 1 - F(\bar{b})$.



(a) Determination of $a(b)$

(b) $a(\bar{b}) = \mathbb{E}[q|q < \bar{b}]$

Figure 4: Illustration of $a(b)$

The role of $a(b)$ in our analysis is reflected in the following lemma. It implies that, to find an industry-optimal signal distribution, it is without loss of generality to focus on incremental benefit functions of the form $c_{a(b),b,v}$.

Lemma 4. *If $c_{a,b,v}$ is feasible, then $c_{a(b),b,v}$ is feasible.*

We are now ready to state our next result.

Proposition 3. *Suppose the value distribution F has a positive and continuous density f over $(0,1)$ such that the hazard rate $\frac{f(x)}{1-F(x)}$ is increasing. Then, $c_{a(\hat{b}),\hat{b},0}$ is feasible if and only if $s \in [\tilde{s}, \bar{s}]$, where $\tilde{s}$ is the unique solution over $(0, \hat{s})$ to the following equation:*[19]

$$\bar{b}(1 - F(\bar{b}))\big[1 - \log(1 - F(\bar{b}))\big] + \tilde{s} = \mu. \tag{8}$$

Moreover, when $s \in [\tilde{s}, \bar{s}]$, $c_{a(\hat{b}),\hat{b},0}$ is the unique industry-optimal incremental benefit function in $\mathcal{U}$.

Under the assumption of increasing hazard rate, Proposition 3 first asks when it is feasible for the industry to fully extract the highest achievable total welfare $\hat{b}$. By Lemma 4, this question is equivalent to the feasibility of $c_{a(\hat{b}),\hat{b},0}$. Proposition 3 shows that it is feasible if and only if the search cost is not too low. Consequently, $c_{a(\hat{b}),\hat{b},0}$

[19] Over $[0, \mu]$, there are two and only two solutions to (8). The obvious one is $s = \mu$. The other appears in $(0, \hat{s})$. See Claim 3 in Appendix D.2.

in this case is industry-optimal. Proposition 3 further strengthens this implication and shows that $c_{a(\hat{b}),\hat{b},0}$ is in fact the unique incremental benefit function in $\mathcal{U}$ that is industry-optimal. That is, there is no other a such that $c_{a,\hat{b},0}$ is feasible. This result is intuitive because $a(\hat{b})$ by construction uniquely maximizes the probability of trade per match for total welfare $\hat{b}$. The highest probability of trade per match means the fewest searches by consumers, thereby minimizing competition among firms.

The analysis of Proposition 3 is a little involved. We use uniform distribution $F \sim U[0,1]$ to explain the ideas. For the uniform distribution, it is easy to calculate $c_F(x) = \frac{1}{2} - x + \frac{1}{2}x^2$ and $\bar{b} = 1 - \sqrt{2s}$. Hence, the search cost $\hat{s}$ that solves $\bar{b} = \frac{1-F(\bar{b})}{f(\bar{b})}$ is $\hat{s} = 0.125$. We can also calculate numerically $\tilde{s} \approx 0.041$ and $\bar{s} \approx 0.296$. Hence, $\tilde{s} < \hat{s} < \bar{s}$ as claimed.

Consider first the special search cost $s = \hat{s}$. Full information disclosure induces equilibrium $(\bar{b}, 0)$. It achieves the highest possible level of total welfare of this market and the industry extracts it all since $\bar{b} = \frac{1-F(\bar{b})}{f(\bar{b})}$. Because $a(\bar{b}) = \mathbb{E}[q|q < \bar{b}]$, we know $c_{a(\bar{b}),\bar{b},0}$, or equivalently $c_{a(\hat{b}),\hat{b},0}$, is feasible by Proposition 1. Clearly, both full information disclosure and $c_{a(\hat{b}),\hat{b},0}$ are industry-optimal. This is an example of multiple optimal signal distributions.

Consider the next case, $s \in (0, \hat{s})$. Full information disclosure still induces an equilibrium, implying $\hat{b} = \bar{b}$. For the uniform distribution, we can also easily verify that $\bar{b}(1 - F(\bar{b})) < \mathbb{E}[q|q < \bar{b}]$. Under $c_{a(\bar{b}),\bar{b},0}$, the left hand side of this inequality is the expected profits π of a matched firm, while the right hand side is simply $a(\bar{b})$. Hence, this inequality simply implies that $c_{a(\bar{b}),\bar{b},0}$ is type II. Proposition 3 shows that it is feasible if and only if the search cost is not sufficiently low and explicitly characterizes the threshold $\tilde{s}$. When $s \geq \tilde{s}$, $c_{a(\bar{b}),\bar{b},0}$, or equivalently $c_{a(\hat{b}),\hat{b},0}$, is clearly industry-optimal. This result goes along with the standard intuition that lower search cost intensifies competition among firms, which makes it harder and eventually impossible for the industry to extract all the surplus at the highest total welfare level. Admittedly, we have mentioned in the discussion of Proposition 2 that this intuition in general may not apply when information also changes with search cost. It still applies in the current particular situation, because $c_{a(\bar{b}),\bar{b},0}$, as search cost decreases, does not change in a way to soften competition. This is suggested by the fact that the probability of trade under $c_{a(\bar{b}),\bar{b},0}$ is $1 - F(\bar{b})$, which is strictly decreasing as s decreases and eventually converges to zero as s goes to zero. Consequently, as search cost decreases, consumers under $c_{a(\bar{b}),\bar{b},0}$ indeed search more as as they do under full information disclosure, which makes the market more competitive.

The third case is $s \in (\hat{s}, \bar{s}]$. In this case, full information disclosure no longer

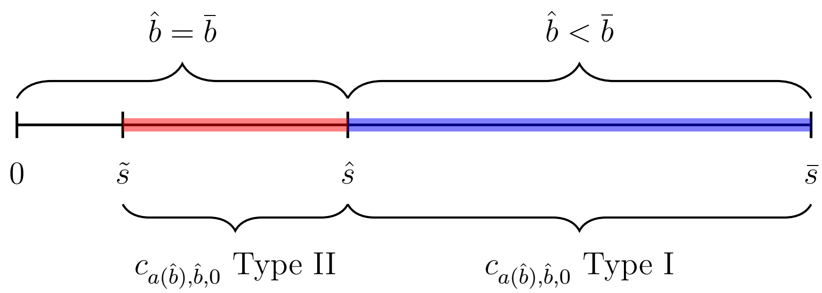


Figure 5: Different cases of $c_{a(\hat{b}),\hat{b},0}$ for the uniform value distribution

induces an equilibrium. Indeed, we can show that $\bar{b}$ is no longer an achievable level of total welfare. Thus, $\hat{b} < \bar{b}$.[20] Moreover, because of uniform distribution again, we can verify $\bar{b}(1 - F(\bar{b})) > \mathbb{E}[q|q < \bar{b}]$. Contrary to the previous case, this inequality implies that $c_{a(\bar{b}),\bar{b},0}$ is type I, which in turn implies that $c_{a(\hat{b}),\hat{b},0}$ is type I too. Hence, for the feasibility of $c_{a(\hat{b}),\hat{b},0}$, only those $\bar{h}(x) \leq c_F(x)$ for $x \in [\hat{b}, 1]$ matter. Similarly as above, we show that these constraints are indeed satisfied. Therefore, $c_{a(\hat{b}),\hat{b},0}$ is feasible and industry-optimal. Figure 5 provides a summary of the above analysis.

What makes the uniform distribution special in the above explanation is that the cutoff search cost $\hat{s}$ for full information disclosure inducing an equilibrium coincides with the cutoff for type switching of $c_{a(\hat{b}),\hat{b},0}$. This is because $\hat{s}$ is also the unique solution to $\bar{b}(1 - F(\bar{b})) = \mathbb{E}[q|q < \bar{b}]$ for this particular distribution. For a general value distribution with increasing hazard rate, these two cutoffs may differ. Additional cases have to be discussed.

In the monopoly pricing setting in Roesler and Szentes (2017), the firm-optimal signal distribution also achieves the highest possible total welfare, all of which is extracted by the firm. But the optimal signal distributions in these two markets are quite different. In the monopoly setting, it is achieved by no information disclosure at all. However, in the current search market, no information disclosure would simply lead to an inactive market. The industry-optimality in Proposition 3 is achieved by a carefully designed conditional unit-elastic signal distribution.

Finally, what is the industry-optimal signal distribution when $s < \tilde{s}$? In this case, because it is impossible for the industry to fully extract the highest achievable total welfare $\hat{b}$, the determination of the industry-optimal signal distribution becomes difficult. It involves a nontrivial trade-off between total welfare and equilibrium price. For example, because $c_{a(\bar{b}),\bar{b},0}$ is no longer feasible, the highest feasible price for total welfare $\bar{b}$ must be strictly less than $\bar{b}$. This simply creates the possibility that a higher

[20] See Claim B.2 in the online appendix.

equilibrium price is achieved at a lower level of total welfare than $\bar{b}$. The next result below states that this is indeed the case if the density of the value distribution is increasing.

Proposition 4. *Suppose the value distribution F has a positive, continuous, and weakly increasing density f over $(0,1)$. Assume $s \in (0, \tilde{s})$. Then, $c_{a(\tilde{b}),\tilde{b},0}$ is the unique industry-optimal incremental benefit function in $\mathcal{U}$, where $\tilde{b}$ is the largest $b \in [\mu - s, \bar{b}]$ such that $c_{a(b),b,0}$ is feasible.*

When f is increasing, Proposition 4 shows that the industry still extracts all the *equilibrium surplus* under the optimal signal distribution when the search cost is below $\tilde{s}$. Because $c_{a(\bar{b}),\bar{b},0}$ is not feasible in this case, we must have $\tilde{b} < \bar{b}$. Therefore, unlike the case when $s \geq \tilde{s}$, the current industry-optimal signal distribution *does not* achieve the highest feasible level of total welfare. As we have explained previously, the problem of achieving the highest total welfare $\bar{b}$ is that it is associated with very small probability of trade per match when s is sufficiently small. This means that consumers search intensively to compare the products across firms, which clearly leads to very competitive market. To soften the competition, it is then necessary to reduce consumers' search incentive. This can only be done by achieving a lower total welfare level, since lower total welfare is associated with larger probability of trade per match and hence less search.[21]

Figure 6 illustrates the optimal industry surplus for the uniform value distribution. Thinking of the search cost s (the vertical axis) as the independent variable, the red curve plots the corresponding optimal industry surplus. Because the industry always extracts all the total welfare under the optimal signal distribution by Propositions 3 and 4, this curve also represents the corresponding total welfare. The curve c_F simply represents $\bar{b}$. When $s \in (\hat{s}, \bar{s}]$, the optimal industry surplus is $\hat{b}$. Because $\hat{b} < \bar{b}$ for search costs in this range, the red curve is below c_F. When $s \in [\tilde{s}, \hat{s}]$, the red curve simply coincides with c_F, since the optimal industry surplus is $\hat{b} = \bar{b}$. When $s < \tilde{s}$, the red curve is below c_F again, because the optimal industry surplus is $\tilde{b}$ from Proposition 4 and it is strictly lower than $\bar{b}$. Overall, the red curve is strictly decreasing in s. This is exactly the result of Proposition 2: as search cost decreases, the optimal industry surplus strictly increases.

[21] The largest probability of trade for total welfare b, i.e., $\frac{\mu - s - a(b)}{b - a(b)}$ is decreasing in b. See part (ii) of 1 in the appendix.

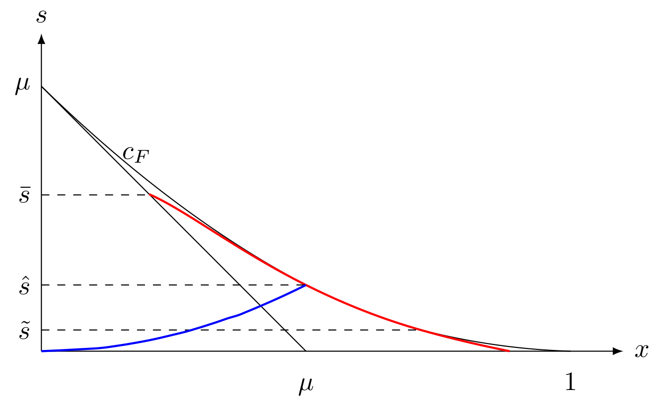


Figure 6: Optimal industry surplus for the uniform value distribution

4.3 Discussions

Full information benchmark Consider a value distribution with an increasing hazard rate again (to guarantee the existence of equilibrium under full information disclosure). As the search cost decreases, the equilibrium consumer surplus is strictly increasing under full information disclosure, while the corresponding equilibrium price and thus industry surplus is strictly decreasing. The latter is illustrated by the blue curve in Figure 6. It is the industry surplus under full information disclosure for uniform value distribution. Dogan and Hu (2022) show that although there is always positive value of information design for consumers as compared to full information disclosure, the value will eventually disappear as the search cost tends to zero. Thus, there is not much room to improve the consumer surplus when the search friction is small. However, Proposition 2 implies that this is exactly the opposite when it comes to the design for the industry. The gap between the optimal industry surplus and the full information benchmark, e.g., the gap between the red and blue curves in Figure 6, strictly increases as the search cost diminishes. In other words, information design for the industry is more valuable compared to the full information benchmark in a market with less search friction.

Total welfare Armstrong and Zhou (2022) study the information design problem in duopolistic competition in a discrete choice model. Under a log-concavity assumption of the distribution of the relative valuation, they find that the industry-optimal signal distribution maximizes the total welfare and it is extracted all by the industry. Our Proposition 3 shows that the same result holds in the current model, provided that the search cost is not too low. Moreover, they also show that the consumer-optimal design typically does not achieve the highest total welfare, and then conclude that the industry-optimal design leads to strictly higher total welfare than the consumer-optimal design in their model. Dogan and Hu (2022) show in the current model that when it comes to consumer-optimal design, there is always a nontrivial trade-off between total welfare and highest consumer surplus. Thus, the total welfare under the consumer-optimal signal distribution is typically not maximized. Therefore, the comparison of total welfare under the industry-optimal and consumer-optimal designs in the current model is also the same as that in Armstrong and Zhou (2022), when the search cost is not too low.

However, when the search cost becomes really small and the density of the value distribution is increasing, Proposition 4 shows that this is no longer the case. The industry-optimal design still guarantees full extraction of equilibrium total welfare by the industry, but the total welfare is not maximized. As we have discussed above, the main reason behind this result is that achieving a lower total welfare than the highest one can induce higher probability of trade per match, which in turn means less search by consumers and thus less competitive market. Clearly, softening competition through the channel of consumers' search behavior is not present in the model of Armstrong and Zhou (2022). Therefore, the result becomes different. Moreover, Claim B.3 in the online appendix shows that, as the search cost converges to zero, the limit of the industry-optimal surplus and thus the total welfare (they are equal by Proposition 4) is strictly bounded above by 1. This can be seen from the uniform distribution example in Figure 6. However, as we have also mentioned above, Dogan and Hu (2022) show that total welfare under the consumer-optimal design must increase to 1 when the search cost converges to zero. Therefore, in a market with very small search friction, it is the consumer-optimal design instead of the industry-optimal design that leads to higher total welfare.

5 Equilibria in Mixed Strategies

In all the previous analysis, we have only considered pure strategy equilibria. Obviously, this restriction automatically rules out those signal distributions that can induce active search only in mixed strategies. Nonetheless, we show here that this restriction is immaterial to the optimal industry surplus. No industry surplus achieved by a mixed strategy equilibrium under a feasible signal distribution can exceed the optimal industry surplus from pure strategy equilibria.

We continue to focus on symmetric equilibria in which all the firms follow the same pricing strategy. As a result, consumers' optimal stopping decision is still a cutoff rule. Formally, a mixed strategy equilibrium with active search under signal distribution G can be characterized by a pair (σ, v), where σ is the firms' mixed strategy over equilibrium signal cutoffs and $v \geq 0$ is the consumer surplus as before. Similarly as in Lemma 1, (σ, v) is an equilibrium if

$$\int_{\text{supp}(\sigma)} c_G(b)\mathrm{d}\sigma(b) = s, \tag{9}$$

and, for all $b \in \text{supp}(\sigma)$,

$$-(b-v)c_G'(b-) \geq -(x-v)c_G'(x-), \ \forall x \in [v,1], \tag{10}$$

where $\text{supp}(\sigma)$ is the support of σ. Condition (9) is a variant of condition (2). It states that the *average incremental gain* from one more search is equal to the cost of one more search.[22] Condition (10) states that every equilibrium signal cutoff $b \in \text{supp}(\sigma)$ must be optimal for the firms. For each such b, condition (10) takes exactly the same form as condition (3). This is because every firm is still competing with the consumers' outside option v. The corresponding expected total welfare of this equilibrium is[23]

$$b_e = \frac{\int_{\text{supp}\sigma}(-c_G'(b-))b\mathrm{d}\sigma(b)}{\int_{\text{supp}\sigma}(-c_G'(b-))\mathrm{d}\sigma(b)}, \tag{11}$$

and thus the expected industry surplus is $b_e - v$.

The following result states that if a feasible signal distribution induces an equilibrium in mixed strategies, then there exists another feasible signal distribution that induces a pure strategy equilibrium with weakly higher industry surplus. In other words, signal distributions that induce equilibria in mixed strategies can not lead to

[22]Given the firms' mixed strategy σ_p over prices, consumers' optimality condition requires

$$v = -s + \int_{\text{supp}(\sigma_p)} \int_{[0,1]} \max\{q-p, v\}\mathrm{d}G(q)\mathrm{d}\sigma_p(p).$$

Condition (9) is obtained by subtracting both sides by v and letting σ be the distribution of $p+v$.

[23]The expected total welfare b_e can be calculated recursively by

$$\begin{aligned} b_e &= -s + \int \big[(1-G(b-))\mathbb{E}_G[q|q \geq b] + G(b-)b_e\big]\mathrm{d}\sigma(b) \\ &= -s + \int \big[c_G(b) + (1-G(b-))b + G(b-)b_e\big]\mathrm{d}\sigma(b) \\ &= \int \big[-c_G'(b-)b + (1+c_G'(b-))b_e\big]\mathrm{d}\sigma(b), \end{aligned}$$

where the last equality comes from equilibrium condition (9). It is then straightforward to obtain (11).

strictly higher industry surplus than the optimal one among those that induce pure strategy equilibria.

Proposition 5. *Suppose (σ, v) is a mixed strategy equilibrium under a feasible signal distribution G. Let b_e be the equilibrium expected total welfare as defined in (11). Then, there exist some a and $b \geq b_e$ such that $c_{a,b,v} \in \mathcal{U}$.*

The proof of this proposition is involved and is in Online Appendix C. The basic idea involves two major steps. In the first step, we show that the result of this proposition is true if σ only randomizes between two signal cutoffs. This is done by explicitly constructing the desired $c_{a,b,v}$. In the second step, we show that if the support of σ contains more than two signal cutoffs, there must exist a new mixed strategy equilibrium under the same signal distribution in which the firms only randomize between two signal cutoffs and the industry surplus is weakly higher. Combining these two steps leads to the desired result.

6 Conclusion

This paper has studied the welfare effect of flexible information design in the consumer search market. We constructed a class of conditional unit-elastic demand signal distributions and showed that this parametric class of signal distributions is rich enough to achieve every equilibrium that can possibly arise under an arbitrary feasible signal distribution. The construction generalizes that in Roesler and Szentes (2017) by incorporating an endogenous outside option and search incentives. The constructed class of signal distributions extends that in Dogan and Hu (2022), which overcomes the limitation that the signal distributions constructed in Dogan and Hu (2022) can only be used to study the consumer-optimal design.

These signal distributions help us understand the welfare limit of the search market. Contrary to the traditional wisdom that less search friction promotes competition between firms, which would reduce the industry surplus, we show that the optimal industry surplus in fact strictly increases as the search cost decreases as long as the information can be adjusted flexibly as the search cost changes. Under certain regularity conditions on the true value distribution, we fully characterize the unique industry-optimal signal distribution. Provided the search cost is not too low, the industry-optimal signal distribution achieves the highest possible total welfare and the industry extracts it all. When the search cost is really low, the industry-optimal signal distribution still leads to full surplus extraction by the industry but the total welfare is not maximized.

An interesting avenue for future research would be to extend the current analysis to the search market where there are only finitely many firms and the consumers' search order is endogenous, as is studied in Zhou (2011) and Armstrong (2017). Such a richer environment would allow us to address interesting questions such as how market size affect the optimal design. Another related and promising question is the design of the pre-match information that can be used to induce different search orders of the consumers, like the one studied in Choi et al. (2018) under exogenous information. We leave this direction of research for future study.

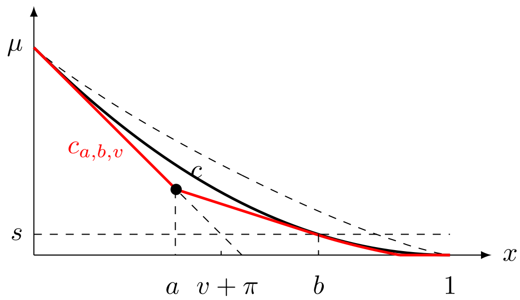
Appendix A Proofs for Section 3

Proof of Proposition 1. First, we show that every $c_{a,b,v}$ induces equilibrium (b, v). Clearly, we have $c_{a,b,v}(b) = s$ by construction. Hence, equilibrium condition (2) for the consumers' search incentive is satisfied. For the firms' pricing incentive in condition (3), observe that if a firm sets cutoff $x \in [b, \bar{x}]$, its expected profits are $-c'_{a,b,v}(x-)(x-v) = -\bar{h}'(x)(x-v) = \frac{\pi}{x-v}(x-v) = \pi$. Hence, no firm has an incentive to deviate from cutoff b to any cutoff $x \in (b, \bar{x}]$. If $c_{a,b,v}$ is type I, i.e., $a \le v + \pi$, deviating to cutoff a, or equivalently charging price $a - v$, yields demand $-c'_{a,b,v}(a-) = 1$. The resulting expected profits are thus $(a-v)\times 1 \le \pi$. Hence, no firm has an incentive to deviate to this cutoff. From panel (a) of Figure 3, it is also clear that deviating to cutoff $x \in [0,1]\backslash(\{a\}\cup[b,\bar{x}])$ can not be profitable. This verifies that (b, v) is an equilibrium when $c_{a,b,v}$ is type I. If $c_{a,b,v}$ is type II, setting cutoff $x \in [v+\pi, \underline{x}]$ yields expected profits $-c'_{a,b,v}(x-)(x-v) = -\underline{h}'(x)(x-v) = \frac{\pi}{x-v}(x-v) = \pi$. Hence, no firm has an incentive to deviate to any cutoff $x \in [v + \pi, \underline{x}]$. From panel (b) of Figure 3, it is also clear that deviating to cutoff $x \in [0, 1]\backslash([v + \pi, \underline{x}] \cup [b, \bar{x}])$ can not be profitable. This verifies that (b, v) is an equilibrium if $c_{a,b,v}$ is type II too.

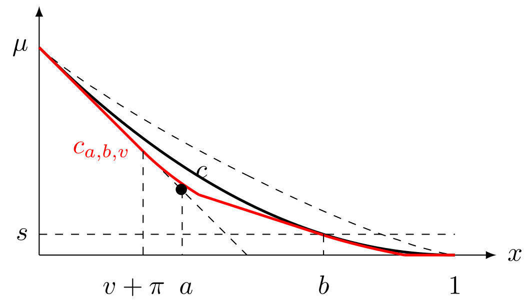
Next, suppose (b, v) is an equilibrium under a feasible signal distribution G. Let $a \equiv \mathbb{E}_G[q|q < b]$. Consider $c_{a,b,v}$. Recall that we construct $\rho = \frac{\mu-s-a}{b-a}$ and $\pi = \rho(b-v)$. Observe that

$$\rho = \frac{\mu - s - a}{b - a} = \frac{\mu - c_G(b) - \mathbb{E}_G[q|q < b]}{b - \mathbb{E}_G[q|q < b]} = 1 - G(b-) = -c'_G(b-).$$

That is, the constructed ρ is just the probability of trade of the equilibrium under G. This implies that the constructed π is also the expected profits of each matched firm



(a) The corresponding $c_{a,b,v}$ is of type I



(b) The corresponding $c_{a,b,v}$ is of type II

Figure 7: Proof of Proposition 1

in the equilibrium under G, and that ℓ by construction is just the left tangent line of c_G at b. For any $x \in [b, 1]$,

$$\bar{h}(x) = \bar{h}(b) + \int_b^x \bar{h}'(\tilde{x})\mathrm{d}\tilde{x} = s - \int_b^x \frac{\pi}{\tilde{x} - v}\mathrm{d}\tilde{x} \leq s + \int_b^x c_G'(\tilde{x}-)\mathrm{d}\tilde{x} = c_G(x),$$

where the inequality comes from the firms' pricing incentive in the equilibrium under G: $-c_G'(\tilde{x})(\tilde{x} - v) \leq \pi$ for all $\tilde{x}$. Hence, we know $c_{a,b,v}(x) \leq c_G(x)$ for all $x \in [b, 1]$.

If $c_{a,b,v}$ is type I, we immediately know that $c_{a,b,v}(x) \leq c_G(x)$ for all $x \in [0, b]$ as well, because $c_{a,b,v}$ coincides with the lower bound c_{F_0} over $[0, a]$ and is the left tangent of c_G over $[a, b]$. Therefore, $c_{a,b,v} \leq c_G \leq c_F$, implying that $c_{a,b,v} \in \mathcal{U}$. See panel (a) of Figure 7 for an illustration.

If $c_{a,b,v}$ is type II, we have for any $x \in [v + \pi, b]$,

$$\begin{aligned}\underline{h}(x) =& \underline{h}(v + \pi) + \int_{v+\pi}^x \underline{h}'(\tilde{x})\mathrm{d}\tilde{x} = c_{F_0}(v + \pi) - \int_{v+\pi}^x \frac{\pi}{\tilde{x} - v}\mathrm{d}\tilde{x} \\ \leq& c_G(v + \pi) + \int_{v+\pi}^x c_G'(\tilde{x}-)\mathrm{d}\tilde{x} = c_G(x),\end{aligned}$$

where the inequality comes from $c_{F_0} \leq c_G$ and $-c_G'(\tilde{x})(\tilde{x} - v) \leq \pi$ for all $\tilde{x}$ again. This implies that $\underline{h}(b) \leq c_G(b) = s$ and $c_{a,b,v}(x) \leq c_G(x)$ for all $x \in [0, b]$. Therefore, $c_{a,b,v} \in \mathcal{U}$. See panel (b) of Figure 7 for an illustration. □

Proof of Lemma 3. Corollary 1 and its proof (Claim A.2 in the online appendix) in Dogan and Hu (2022) show that there exists $\bar{s} \in (0, \mu)$ such that the following three statements are equivalent: (i) (F, s) admits active search; (ii) $c^s_{0,\mu-s,0} \in \mathcal{C}_F$; and (iii) $s \in (0, \bar{s}]$. Thus, in fact, we can give an expression of $\bar{s}$: $\bar{s} = \max\{s \,|\, c^s_{0,\mu-s,0} \in \mathcal{C}_F\}$. □

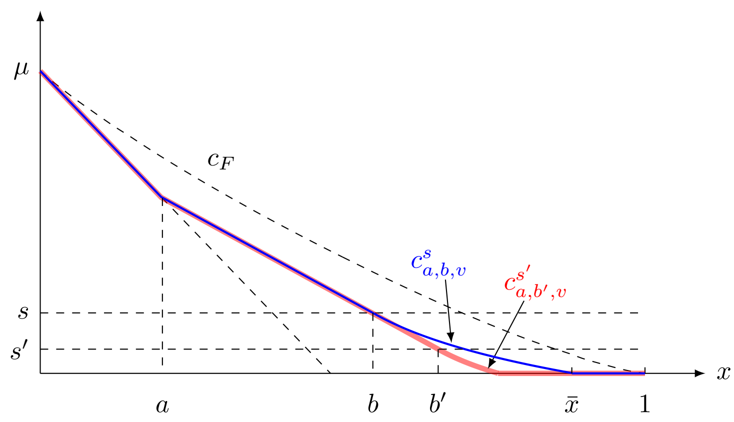


Figure 8: Proof of Proposition 2

Appendix B Proof of Proposition 2

Proof of Proposition 2. To avoid confusion, we explicitly add superscript s and write $c^s_{a,b,v}$ to mean that this incremental benefit function is constructed for search cost s. We also add s as an explicit argument to both $\underline{h}$ and $\bar{h}$.

Consider $0 < s' < s \le \bar{s}$. To show the desired result, it suffices to show that if $c^s_{a,b,v}$ is feasible, then $c^{s'}_{a,b',v}$ is also feasible, where $b' = b + \frac{(s-s')(b-a)}{(\mu-s-a)}$. Figure 8 illustrates how this b' is obtained. It is the intersection of the horizontal line of value s' and the extension of the straight line segment connecting $(a, \mu - a)$ and (b, s).

By construction, the equilibrium probabilities of trade under $c^s_{a,b,v}$ and $c^{s'}_{a,b',v}$ are the same in their respective markets: $\rho = \frac{b-a}{\mu-s-a} = \frac{b'-a}{\mu-s'-a}$. Let $\pi = \rho(b - v)$ (respectively, $\pi' = \rho(b' - v)$) be each firm's expected profits under $c^s_{a,b,v}$ (respectively, $c^{s'}_{a,b',v}$) in the market with search cost s (respectively, s'). Clearly $\pi' > \pi$. Recall that $\ell(x) = s - \rho(x - b)$. It can also be written as $\ell(x) = s' - \rho(x - b')$. For any $x \in [b, b']$, we have $\ell(x) \le \bar{h}(x; a, b, v, s)$. For $x \in [b', 1]$, we have

$$\bar{h}(x; a, b', v, s') = s' - \int_{b'}^{x} \frac{\pi'}{\tilde{x} - v} \mathrm{d}\tilde{x} \le \bar{h}(b'; a, b, v, s) - \int_{b'}^{x} \frac{\pi}{\tilde{x} - v} \mathrm{d}\tilde{x} = \bar{h}(x; a, b, v, s),$$

where the inequality comes from $\ell(b') = s' \le \bar{h}(b'; a, b, v, s)$ and $\pi' > \pi$. Therefore, $c^{s'}_{a,b',v} \le c^s_{a,b,v}$ over $[b, 1]$.

If $c^s_{a,b,v}$ is type I, we know $c^{s'}_{a,b',v}$ is type I too, since $v + \pi' > v + \pi \ge a$. Because they coincide over $[0, b]$, above analysis immediately implies that $c^{s'}_{a,b',v} \le c^s_{a,b,v}$ over the whole interval $[0, 1]$. Therefore, $c^{s'}_{a,b',v}$ is feasible. This case is illustrated in Figure 8. The blue curve represents the given $c^s_{a,b,v}$, while the red curve is the newly constructed $c^{s'}_{a,b',v}$.

If $c^s_{a,b,v}$ is type II and $c^{s'}_{a,b',v}$ is type I, then $c^{s'}_{a,b',v}(x) = \ell(x) \leq \max\{\underline{h}(x; a, b, v, s), \ell(x)\} = c^s_{a,b,v}(x)$ for $x \in [a, b]$. Therefore, $c^{s'}_{a,b',v} \leq c^s_{a,b,v}$, implying that $c^{s'}_{a,b',v}$ is feasible too. Finally, if both $c^s_{a,b,v}$ and $c^{s'}_{a,b',v}$ are type II, we have, for all $x \in [\pi' + v, b]$,

$$\begin{aligned}\underline{h}(x; a, b', v, s') &= \mu - \pi' - v - \int_{\pi'+v}^{x} \frac{\pi'}{\tilde{x} - v} \mathrm{d}\tilde{x} \\ &\leq \underline{h}(\pi' + v; a, b, v, s) - \int_{\pi'+v}^{x} \frac{\pi}{\tilde{x} - v} \mathrm{d}\tilde{x} = \underline{h}(x; a, b, v, s),\end{aligned}$$

where the inequality comes from $\mu - \pi' - v \leq \underline{h}(\pi' + v; a, b, v, s)$ and $\pi' > \pi$. Thus, $c^{s'}_{a,b',v}(x) = \max\{\underline{h}(x; a, b', v, s'), \ell(x)\} \leq \max\{\underline{h}(x; a, b, v, s), \ell(x)\} = c^s_{a,b,v}(x)$ for all $x \in [\pi' + v, b]$. Therefore, $c^{s'}_{a,b',v} \leq c^s_{a,b,v}$, implying that $c^{s'}_{a,b',v}$ is feasible as well. □

Appendix C Proof of Lemma 4

For any b, let $\rho(b) \equiv \frac{\mu - s - a(b)}{b - a(b)}$ be the highest feasible probability of trade for b. We will also use this notation in later proofs.

Proof. Let $\pi \equiv (b - v)\frac{\mu - s - a}{b - a}$ be each firm's expected profits under $c_{a,b,v}$ and $\pi' \equiv (b - v)\rho(b)$ be those under $c_{a(b),b,v}$. Because $\rho(b) \geq \frac{\mu - s - a}{b - a}$ by definition, $\pi' \geq \pi$. Hence, for all $x \in [b, 1]$,

$$\bar{h}(x; a(b), b, v) = s - \pi' \log \frac{x - v}{b - v} < s - \pi \log \frac{x - v}{b - v} = \bar{h}(x; a, b, v) \leq c_F(x), \tag{12}$$

where the last inequality comes from the feasibility of $c_{a,b,v}$.

On the one hand, if $\pi' + v \geq a(b)$, we know that $c_{a(b),b,v}$ is type I. Inequality (12) then implies the feasibility of $c_{a(b),b,v}$. On the other hand, if $\pi' + v < a(b)$, we know $\pi + v \leq \pi' + v < a(b) \leq a$, implying that both $c_{a,b,v}$ and $c_{a(b),b,v}$ are type II. Moreover, for all $x \in [\pi' + v, b]$,

$$\underline{h}(x; a(b), b, v) = \mu - \pi' - v - \pi' \log \frac{x - v}{\pi'} < \mu - \pi - v - \pi \log \frac{x - v}{\pi} = \underline{h}(x; a, b, v). \tag{13}$$

Because $c_{a,b,v}$ is feasible, we then know $\underline{h}(x; a(b), b, v) \leq c_F(x)$ for $x \in [\pi' + v, b]$ and $\underline{h}(b; a(b), b, v) \leq s$. These inequalities, together with (12), imply the feasibility of $c_{a(b),b,v}$.

Notice that this result holds for general value distribution F. □

Appendix D Proof of Proposition 3

D.1 General properties of feasibility

In this subsection, we provide some general properties of feasibility that do not rely on the increasing hazard rate assumption. These properties will also be used in later analysis.

The following claim lists some simple properties of $a(b)$ that will be used frequently in the following analysis.

Claim 1. *(i) For every b, there exists a unique $\hat{x} \in [0, b]$ such that the straight line $\ell(x) \equiv s - \rho(b)(x - b)$ is tangent to c_F at $\hat{x}$.*

(ii) $a(b)$ is increasing, while $\rho(b)$ and $b\rho(b)$ are decreasing.

(iii) $a(\bar{b}) = \mathbb{E}[q|q < \bar{b}]$ and $\rho(\bar{b}) = 1 - F(\bar{b}-)$.

Proof. All the results are obvious from Figure 9, and thus the details are omitted. □

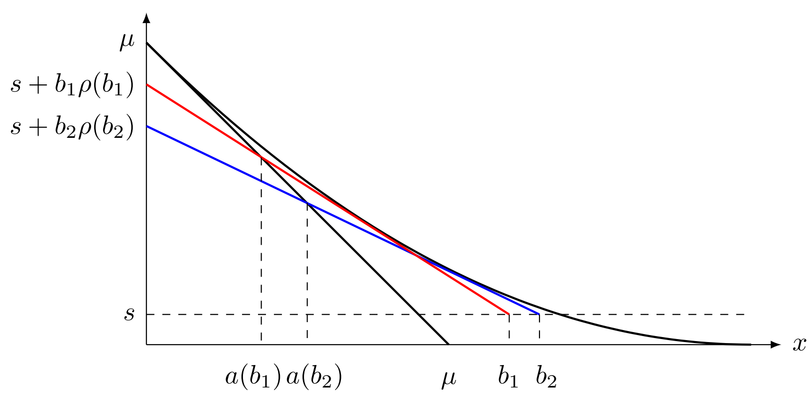


Figure 9: Proof of Claim 1

Claim 2. *For any $s \in (0, \bar{s}]$, $c_{a(\hat{b}),\hat{b},0}$ is feasible if one of the following two conditions is satisfied:*

(i) it is type I;

(ii) it is type II and $\underline{h}(x; a(\hat{b}), \hat{b}, 0) \leq c_F(x)$ for all $x \in [\pi, \hat{b}]$, where $\pi = \hat{b}\rho(\hat{b})$.

Proof. Because $\hat{b}$ is an achievable level of total welfare, there exist some a and v such that $c_{a,\hat{b},v}$ is feasible. By Lemma 4, we know $c_{a(\hat{b}),\hat{b},v}$ is feasible. Hence, for any

$x \in [\hat{b}, 1]$,

$$\bar{h}(x; a(\hat{b}), \hat{b}, 0) = s - \int_{\hat{b}}^{x} \frac{\hat{b}\rho(\hat{b})}{t} \mathrm{d}t \leq s - \int_{\hat{b}}^{x} \frac{(\hat{b}-v)\rho(\hat{b})}{t-v} \mathrm{d}t = \bar{h}(x; a(\hat{b}), \hat{b}, v) \leq c_F(x). \tag{14}$$

If $c_{a(\hat{b}),\hat{b},0}$ is type I, (14) implies that it is feasible. This proves part (i). For part (ii), suppose $c_{a(\hat{b}),\hat{b},0}$ is type II, and $\underline{h}(x; a(\hat{b}), \hat{b}, 0) \leq c_F(x)$ for $x \in [\pi, \hat{b}]$. For $c_{a(\hat{b}),\hat{b},0}$ to be feasible, we only need to verify $\underline{h}(\hat{b}; a(\hat{b}), \hat{b}, 0) \leq s$. If $\hat{b} = \bar{b}$, this clearly holds, since $c_F(\bar{b}) = s$. Assume $\hat{b} < \bar{b}$. By part (i) of Claim 1, the straight line $\ell(x) \equiv s - \rho(\hat{b})(x - \hat{b})$ intersects c_F at some $\hat{x} \in (a(\hat{b}), \hat{b}) \subset [\pi, \hat{b}]$. Hence $\underline{h}(\hat{x}; a(\hat{b}), \hat{b}, 0) \leq \ell(\hat{x})$. Consequently,

$$\underline{h}(\hat{b}; a(\hat{b}), \hat{b}, 0) = \underline{h}(\hat{x}; a(\hat{b}), \hat{b}, 0) - \int_{\hat{x}}^{\hat{b}} \frac{\hat{b}\rho(\hat{b})}{t} \mathrm{d}t < \ell(\hat{x}) - \int_{\hat{x}}^{\hat{b}} \rho(\hat{b}) \mathrm{d}t = \ell(\hat{b}) = s,$$

proving part (ii). □

D.2 (In)feasibility of $c_{a(\hat{b}),\hat{b},0}$

In this and the next subsections, we maintain the assumption that F has a continuous and strictly positive density f with increasing hazard rate. This subsection characterizes the (in)feasibility of $c_{a(\hat{b}),\hat{b},0}$ for different search costs. The next section proves the uniqueness of the optimal signal distribution.

Claim 3. *Equation (8) has a unique solution $\tilde{s} \in (0, \mu)$. Moreover,*

(i) $\underline{h}(\bar{b}; a(\bar{b}), \bar{b}, 0) \leq s$ if and only if $s \geq \tilde{s}$; and

(ii) if $s < \tilde{s}$, then both $\bar{b} > \frac{1-F(\bar{b})}{f(\bar{b})}$ and $\bar{b}(1 - F(\bar{b})) < \mathbb{E}[q|q < \bar{b}]$ hold.

Proof. Define $\phi : [0, 1] \to \mathbb{R}$ as $\phi(x) \equiv x(1-F(x))[1 - \log(1 - F(x))] + c_F(x) - \mu$. It is easy to calculate $\phi'(x) = f(x) \log(1-F(x)) \left[x - \frac{1-F(x)}{f(x)}\right]$. Because $\frac{1-F(x)}{f(x)}$ is decreasing, there exists a unique $\hat{x} \in (0, 1)$ such that $\hat{x} = \frac{1-F(\hat{x})}{f(\hat{x})}$. Moreover, $x < \frac{1-F(x)}{f(x)}$ if $x < \hat{x}$, and $x > \frac{1-F(x)}{f(x)}$ if $x > \hat{x}$. Therefore, $\phi'(x) > 0$ if $x < \hat{x}$ and $\phi'(x) < 0$ if $x > \hat{x}$. Because $\phi(0) = 0$, we know $\phi(x) > 0$ if $x \leq \hat{x}$. Because $\phi(1) = -\mu < 0$, we know there exists a unique $\tilde{x} \in (\hat{x}, 1)$ such that $\phi(\tilde{x}) = 0$. Then, $\tilde{s} \equiv c_F(\tilde{x})$ is the unique positive solution to (8). When $s \geq \tilde{s}$, the corresponding $\bar{b}$ satisfies $\bar{b} \leq \tilde{x}$. By the above analysis, we know $\phi(\bar{b}) \geq 0$. Using part (iii) of Claim 1, we observe that this inequality is equivalent to $\underline{h}(\bar{b}; a(\bar{b}), \bar{b}, 0) \leq s$. Similarly, when $s < \tilde{s}$, we have $\underline{h}(\bar{b}; a(\bar{b}), \bar{b}, 0) > s$. This proves part (i).

Suppose $s < \tilde{s}$. The corresponding $\bar{b}$ satisfies $\bar{b} > \tilde{x} > \hat{x}$. Thus, $\bar{b} > \frac{1-F(\bar{b})}{f(\bar{b})}$. For the second inequality, define $\eta : [0, 1] \to \mathbb{R}$ as $\eta(x) \equiv x(1 - F(x))F(x) - \int_0^x tf(t)\mathrm{d}t$. With some algebra, we can write $\eta(x) = x(1 - F(x))(1 + F(x)) + c_F(x) - \mu$. Because $1 + F(x) < 1 - \log(1 - F(x))$ for $x \in (0, 1)$, we know $\phi(x) > \eta(x)$ for $x \in (0, 1)$. This implies that $\eta(x) < 0$ if $x > \tilde{x}$. Consequently, $\eta(\bar{b}) < 0$, since $\bar{b} > \tilde{x}$. Equivalently, $\bar{b}(1 - F(\bar{b})) < \mathbb{E}[q|q < \bar{b}]$. This proves part (ii). □

Claim 4. *For any $s \in [\tilde{s}, \bar{s}]$, $c_{a(\hat{b}),\hat{b},0}$ is feasible.*

Proof. We discuss two cases.

Case (i): $\bar{b}(1 - F(\bar{b})) \geq \mathbb{E}[q|q < \bar{b}]$.

By part (iii) of Claim 1, we can write $\bar{b}(1 - F(\bar{b})) \geq \mathbb{E}[q|q < \bar{b}]$ as $\bar{b}\rho(\bar{b}) \geq a(\bar{b})$. By part (ii) of Claim 1 and the fact that $\hat{b} \leq \bar{b}$, we know $a(\hat{b}) \leq a(\bar{b}) \leq \bar{b}\rho(\bar{b}) \leq \hat{b}\rho(\hat{b})$. Therefore, $c_{a(\hat{b}),\hat{b},0}$ is type I, and hence is feasible by Claim 2.

Case (ii): $\bar{b}(1 - F(\bar{b})) < \mathbb{E}[q|q < \bar{b}]$.

Let $\pi \equiv \bar{b}\rho(\bar{b})$ for short. We have $\pi < a(\bar{b})$. We first show $\underline{h}(x; a(\bar{b}), \bar{b}, 0) \leq c_F(x)$ for all $x \in [\pi, \bar{b}]$. When $x = \pi$, we know $\underline{h}(\pi; a(\bar{b}), \bar{b}, 0) = \mu - \pi < c_F(\pi)$, where the strict inequality comes from full support of F. Suppose, by contradiction, $\underline{h}(\tilde{x}; a(\bar{b}), \bar{b}, 0) > c_F(\tilde{x})$ for some $\tilde{x} \in (\pi, \bar{b}]$. Then, there must exist $\hat{x} \in [\pi, \tilde{x})$ such that $\underline{h}_x(\hat{x}; a(\bar{b}), \bar{b}, 0) > c'_F(\hat{x})$. That is, $-\frac{\pi}{\hat{x}} > -(1-F(\hat{x}))$, or equivalently $\pi < \hat{x}(1-F(\hat{x}))$. Because $\frac{1-F(x)}{f(x)}$ is decreasing, we know $\pi < x(1-F(x))$ for all $x \in [\hat{x}, \bar{b})$. Equivalently, $\underline{h}_x(x; a(\bar{b}), \bar{b}, 0) > c'_F(x)$ for all $x \in [\hat{x}, \bar{b})$. This implies $\underline{h}(\bar{b}; a(\bar{b}), \bar{b}, 0) = \underline{h}(\tilde{x}; a(\bar{b}), \bar{b}, 0)+ \int_{\tilde{x}}^{\bar{b}} \underline{h}(x; a(\bar{b}), \bar{b}, 0)\mathrm{d}x > c_F(\tilde{x}) + \int_{\tilde{x}}^{\bar{b}} c'_F(x)\mathrm{d}x = c_F(\bar{b})$. But this contradicts part (i) of Claim 3, because $s \geq \tilde{s}$. Therefore, we must have $\underline{h}(x; a(\bar{b}), \bar{b}, 0) \leq c_F(x)$ for $x \in [\pi, \bar{b}]$.

If $\hat{b} = \bar{b}$, then $c_{a(\hat{b}),\hat{b},0}$ is type II. The above analysis and Claim 2 guarantee that it is feasible. Assume $\hat{b} < \bar{b}$. If $c_{a(\hat{b}),\hat{b},0}$ is type I, we know it is feasible by Claim 2. Suppose it is type II. Let $\pi' \equiv \hat{b}\rho(\hat{b})$. By part (ii) of Claim 1, we have $\pi' \geq \pi$. Therefore, for any $x \in [\pi', \hat{b}] \subset [\pi, \bar{b}]$, we have

$$\underline{h}(x; a(\hat{b}), \hat{b}, 0) = \mu - \pi' - \pi' \log \frac{x}{\pi'} \leq \mu - \pi - \pi \log \frac{x}{\pi} = \underline{h}(x; a(\bar{b}), \bar{b}, 0) \leq c_F(x).$$

By Claim 2, $c_{a(\hat{b}),\hat{b},0}$ is feasible. □

Claim 5. *For every $s \in (0, \tilde{s})$, $c_{a(\hat{b}),\hat{b},0}$ is not feasible.*

Proof. By part (ii) of Claim 3, we know $\hat{b} = \bar{b}$ and $c_{a(\bar{b}),\bar{b},0}$ is type II. Part (i) of Claim 3 then implies that $c_{a(\bar{b}),\bar{b},0}$ is not feasible. □

D.3 Uniqueness

Claim 6. *Suppose $s \in [\tilde{s}, \bar{s}]$ and $\hat{b} < \bar{b}$.*

(i) If $c_{a(\hat{b}),\hat{b},0}$ is type I, at least one of the feasibility constraints $\bar{h}(x; a(\hat{b}), \hat{b}, 0) \leq c_F(x)$ for $x \in [\hat{b}, 1]$ must be binding.

(ii) If $c_{a(\hat{b}),\hat{b},0}$ is type II, at least one of the feasibility constraints $\bar{h}(x; a(\hat{b}), \hat{b}, 0) \leq c_F(x)$ for $x \in [\hat{b}, 1]$ and $\underline{h}(x; a(\hat{b}), \hat{b}, 0) \leq c_F(x)$ for $x \in [\hat{b}\rho(\hat{b}),\, \hat{b}]$ must be binding.

Proof. Part (i): $c_{a(\hat{b}),\hat{b},0}$ is type I.

Suppose, by contradiction, $\bar{h}(x; a(\hat{b}), \hat{b}, 0) < c_F(x)$ for all $x \in [\hat{b}, 1]$. By uniform continuity, there exists $b_1 \in (\hat{b}, \bar{b}]$ such that for all $b \in [\hat{b}, b_1]$, $\bar{h}(x; a(b), b, 0) \leq c_F(x)$ for all $x \in [b, 1]$.

If $\hat{b}\rho(\hat{b}) > a(\hat{b})$, we can choose $b \in (\hat{b}, b_1)$ such that $b\rho(b) > a(b)$. This implies that $c_{a(b),b,0}$ is type I and is feasible. This contradicts the fact that $\hat{b}$ is the highest possible level of total welfare.

Suppose now that $\hat{b}\rho(\hat{b}) = a(\hat{b})$. Note that for any $b > \hat{b}$, we know $b\rho(b) < \hat{b}\rho(\hat{b}) = a(\hat{b}) < a(b)$ by part (ii) of Claim 1. Hence $c_{a(b),b,0}$ is type II. Pick $x_1 \in (0,\, \hat{b}\rho(\hat{b}))$. Pick $b_2 \in (\hat{b}, b_1]$ such that $b_2\rho(b_2) \geq x_1$. Consider the function $\kappa : [x_1, b_2] \times [\hat{b}, b_2] \to \mathbb{R}$ as

$$\kappa(x, b) \equiv \begin{cases} \mu - x, & \text{if } x_1 \leq x \leq b\rho(b), \\ \underline{h}(x; a(b), b, 0), & \text{if } b\rho(b) < x \leq b_2. \end{cases}$$

Consider $b = \hat{b}$. Because F has full support, we have $\kappa(x, \hat{b}) = \mu - x < c_F(x)$ for $x \in [x_1, \hat{b}\rho(\hat{b})]$. For $x \in (\hat{b}\rho(\hat{b}), b_2]$, we have $\kappa(x, \hat{b}) = \underline{h}(x; a(\hat{b}), \hat{b}, 0) < s - \rho(\hat{b})(x - \hat{b}) \leq c_F(x)$. This implies that $\kappa(x, \hat{b}) < c_F(x)$ for all $x \in [x_1, b_2]$ and $\kappa(x, \hat{b}) < s$ for all $x \in [\hat{b}, b_2]$. By uniform continuity of κ, we know there exists $b \in (\hat{b}, b_2]$ such that $\kappa(x, b) \leq c_F(x)$ for all $x \in [x_1, b_2]$ and $\kappa(b, b) < s$. Therefore, we have $\underline{h}(x; a(b), b, 0) \leq c_F(x)$ for $x \in [b\rho(b), b] \subset [x_1, b_2]$ and $\underline{h}(b; a(b), b, 0) \leq s$. Because $b \leq b_1$, we also know $\bar{h}(x; a(b), b, 0) \leq c_F(x)$ for $x \in [b, 1]$ from the above analysis. Therefore, $c_{a(b),b,0}$ is feasible. This again contradicts the assumption that $\hat{b}$ is the highest possible level of total welfare.

Part (ii): $c_{a(\hat{b}),\hat{b},0}$ is type II.

Applying a similar argument as in part (i) for the case $\hat{b}\rho(\hat{b}) = a(\hat{b})$, we can show that if none of the feasibility constraints is binding, then there must exist $b > \hat{b}$ such that $c_{a(b),b,0}$ is feasible. This again contradicts the definition of $\hat{b}$. $\square$

Claim 7. *For $s \in [\tilde{s}, \bar{s}]$, $c_{a(\hat{b}),\hat{b},0}$ is the unique industry-optimal signal distribution in $\mathcal{U}$.*

Proof. Because $c_{a(\hat{b}),\hat{b},0}$ is feasible by Claim 4 and the industry obtains the highest possible surplus in this market, it is clear that $c_{a(\hat{b}),\hat{b},0}$ is industry-optimal. To show that it is the unique industry-optimal one in $\mathcal{U}$, we only need to verify that $c_{a,\hat{b},0}$ is not feasible for any $a > a(\hat{b})$. We discuss two cases.

Case (i): $\hat{b} = \bar{b}$.

Consider any $a < a(\bar{b})$. We have

$$\bar{h}_x(\bar{b}; a, \bar{b}, 0) = -\frac{\bar{b}\frac{\mu-s-a}{\bar{b}-a}}{\bar{b}} = -\frac{\mu - s - a}{\bar{b} - a} > -\rho(\bar{b}) = -(1 - F(\bar{b})) = c'_F(\bar{b}).$$

Because F is continuous, we have $\bar{h}_x(x; a, \bar{b}, 0) > c'_F(x)$ over some interval $[\bar{b}, \hat{x}]$. Because $\bar{h}(\bar{b}; a, \bar{b}, 0) = c_F(\bar{b})$, we have $\bar{h}(x; a, \bar{b}, 0) > c_F(x)$ for $x \in (\bar{b}, \hat{x}]$. This shows that $c_{a,\bar{b},0}$ is not feasible.

Case (ii): $\hat{b} < \bar{b}$.

Suppose $c_{a(\hat{b}),\hat{b},0}$ is type I. By Claim 6, there exists $\hat{x} \in [\hat{b}, 1]$ such that $\bar{h}(\hat{x}; a(\hat{b}), \hat{b}, 0) = c_F(\hat{x})$. By the same argument as in (12), we know $\bar{h}(\hat{x}; a, \hat{b}, 0) > \bar{h}(\hat{x}; a(\hat{b}), \hat{b}, 0) = c_F(\hat{x})$ for $a > a(\hat{b})$, showing that $c_{a,\hat{b},0}$ is not feasible for any $a > a(\hat{b})$.

Suppose $c_{a(\hat{b}),\hat{b},0}$ is type II. If there exists $\hat{x} \in [\hat{b}, 1]$ such that $\bar{h}(\hat{x}; a(\hat{b}), \hat{b}, 0) = c_F(\hat{x})$, then using the same argument as above, we know $c_{a,\hat{b},0}$ is not feasible for any $a > a(\hat{b})$. If there is no such $\hat{x}$, we know there exists $\tilde{x} \in [\hat{b}\rho(\hat{b}), \hat{b}]$ such that $\underline{h}(\tilde{x}; a(\hat{b}), \hat{b}, 0) = c_F(\tilde{x})$ by Claim 6. For any $a > a(\hat{b})$, we know $\hat{b}\frac{\mu-s-a}{\hat{b}-a} < \hat{b}\rho(\hat{b}) < a(\hat{b}) < a$. Hence, $c_{a,\hat{b},0}$ is type II. By the same argument as in (13), we know $\underline{h}(\tilde{x}; a, \hat{b}, 0) > \underline{h}(\tilde{x}; a(\hat{b}), \hat{b}, 0) = c_F(\tilde{x})$, showing that $c_{a,\hat{b},0}$ is not feasible. $\square$

Claims 4, 5, and 7 together prove Proposition 3.

Appendix E Proof of Proposition 4

In this section, we maintain the assumption that density f is increasing. The proof of Proposition 4 requires a series of claims.

Claim 8. *For all $x \in [0, 1]$, $\mu(1 - F(x))^2 \geq c_F(x)$.*

Proof. Define $\phi(x) \equiv \mu(1-F(x))^2 - c_F(x)$. Note that $\phi(0) = \phi(1) = 0$. Because f is increasing and $\phi'(x) = 2\mu(1-F(x))(\frac{1}{2\mu} - f(x))$, we know ϕ is increasing first and then decreasing. Therefore, $\phi(x) \geq 0$, or equivalently $\mu(1-F(x))^2 \geq c_F(x)$. $\square$

Claim 9. *Consider $s \in (0, \mu)$. The maximal feasible probability of trade function $\rho(b)$ is differentiable. Moreover, for any $b \in (\mu - s,\, \bar{b})$, we have $(\mu - s)\rho'(b) + 1 < 0$.*

Proof. By part (i) of Claim 1, for any $b \in (\mu - s,\, \bar{b})$, the straight line $\ell(x) \equiv s - \rho(b)(x-b)$ is tangent to c_F at some $x(b) \in (0, \bar{b})$. Therefore, we have $\rho(b) = -c_F'(x(b))$ and for all b,

$$c'(x(b))(b - x(b)) + c(x(b)) = s.$$

Because f is continuous, by the implicit function theorem, we know $x(b)$ is differentiable and

$$x'(b) = \frac{c_F'(x(b))^2}{c_F''(x(b))(c_F(x(b)) - s)}.$$

Because $\rho(b) = -c_F'(x(b))$, we know $\rho(b)$ is differentiable too. Moreover,

$$\rho'(b) = -c_F''(x(b))x'(b) = -\frac{c_F'(x(b))^2}{c_F(x(b)) - s} \leq -\frac{c_F(x(b))}{\mu(c_F(x(b)) - s)},$$

where the inequality comes from Claim 8. Consequently,

$$(\mu - s)\rho'(b) + 1 \leq -\frac{(\mu - s)c_F(x(b))}{\mu(c_F(x(b)) - s)} + 1 = \frac{s(c_F(x(b)) - \mu)}{\mu(c_F(x(b)) - s)} < 0. \qquad \square$$

Claim 10. *Suppose $s \in (0, \tilde{s})$. Let $\tilde{b}$ be the largest $b \in [\mu - s,\, \bar{b}]$ such that $c_{a(b),b,0}$ is feasible. Then $c_{a(\tilde{b}),\tilde{b},0}$ is type II. Moreover, at least one of the feasibility constraints $\underline{h}(x; a(\tilde{b}), \tilde{b}, 0) \leq c_F(x)$ for $x \in [\tilde{b}\rho(\tilde{b}), \tilde{b}]$ is binding.*

Proof. By Proposition 3, we know $\tilde{b} < \hat{b} = \bar{b}$. If $c_{a(\tilde{b}),\tilde{b},0}$ is type I, we can apply the same argument as in the proof of Claim 6 to show that there must exist $b' \in (\tilde{b}, \bar{b}]$ such that either $c_{a(b'),b',0}$ is type I or it is type II and $\underline{h}(x; a(b'), b', 0) \leq c_F(x)$ for $x \in [b'\rho(b'), b']$. By Claim 2, such $c_{a(b'),b',0}$ is feasible. This contradicts the assumption that $\tilde{b}$ is the largest $b \in [\mu - s,\, \bar{b}]$ such that $c_{a(b),b,0}$ is feasible. Hence, $c_{a(\tilde{b}),\tilde{b},0}$ is type II.

We can then apply the same argument again as in the proof of Claim 6 to show that, if none of the feasibility constraints $\underline{h}(x; a(\tilde{b}), \tilde{b}, 0) \leq c_F(x)$ for $x \in [\tilde{b}\rho(\tilde{b}), \tilde{b}]$ is binding, then there exists $b' \in (\tilde{b}, \bar{b}]$ such that $\underline{h}(x; a(b'), b', 0) \leq c_F(x)$ for $x \in [b'\rho(b'), b']$. By Claim 2, $c_{a(b'),b',0}$ is feasible. This contradicts again the definition of $\tilde{b}$. $\square$

Proof of Proposition 4. To show the optimality of $c_{a(\tilde{b}),\tilde{b},0}$, we need to verify that there is no $a \in [0, \mu - s)$, $b \in (\tilde{b}, \bar{b}]$ and $v \in [0, b - \tilde{b}]$ such that $c_{a,b,v}$ is feasible. By Lemma 4, it suffices to show that, for any $b \in (\tilde{b}, \bar{b}]$ and $v \in [0, b - \tilde{b}]$, $c_{a(b),b,v}$ is not feasible.

Because $(\mu - s)\rho'(b) + 1 < 0$ by Claim 9, $\rho'(b) < 0$ and $\tilde{b} \geq \mu - s$, we know $\tilde{b}\rho'(b) + 1 < 0$. Thus, for any $b \in (\tilde{b}, \bar{b}]$, we know $\tilde{b}\rho(b) + (b - \tilde{b}) < \tilde{b}\rho(\tilde{b}) < a(\tilde{b}) < a(b)$, where the second inequality comes from the fact that $c_{a(\tilde{b}),\tilde{b},0}$ is type II by Claim 10 and the last inequality comes from part (ii) of Claim 1. This implies that $c_{a(b),b,b-\tilde{b}}$ is type II.

Let $\hat{x} \in [\tilde{b}\rho(\tilde{b}), \tilde{b}]$ be such that $\underline{h}(\hat{x}; a(\tilde{b}), \tilde{b}, 0) = c_F(\hat{x})$. Its existence is claimed by Claim 10. Note that for any $b > \tilde{b}$, $\hat{x} \in [\tilde{b}\rho(\tilde{b}), \tilde{b}] \subset [\tilde{b}\rho(b) + (b - \tilde{b}), b]$. Thus, $\hat{x}$ is in the relevant domain of $\underline{h}(\,\cdot\,; a(b), b, b - \tilde{b})$. Holding $\hat{x}$ and $\tilde{b}$ fixed, the derivative of $b \mapsto \underline{h}(\hat{x}; a(b), b, b - \tilde{b})$ is

$$\frac{\tilde{b}\rho(b)}{\hat{x} - (b - \tilde{b})} - 1 - \tilde{b}\rho'(b)\log\frac{\hat{x} - (b - \tilde{b})}{\tilde{b}\rho(b)}.$$

As above, since $\tilde{b}\rho'(b) + 1 < 0$, we know $-\tilde{b}\rho'(b) > 1$. Hence, we have

$$\frac{\tilde{b}\rho(b)}{\hat{x} - (b - \tilde{b})} - 1 - \tilde{b}\rho'(b)\log\frac{\hat{x} - (b - \tilde{b})}{\tilde{b}\rho(b)} > \frac{\tilde{b}\rho(b)}{\hat{x} - (b - \tilde{b})} - 1 + \log\frac{\hat{x} - (b - \tilde{b})}{\tilde{b}\rho(b)} > 0.$$

This implies that $\underline{h}(\hat{x}; a(b), b, b - \tilde{b}) > \underline{h}(\hat{x}; a(\tilde{b}), \tilde{b}, 0) = c_F(\hat{x})$ for all $b \in (\tilde{b}, \bar{b}]$. Therefore, $c_{a(b),b,b-\tilde{b}}$ is not feasible for any $b \in (\tilde{b}, \bar{b}]$.

Next, consider any $b \in (\tilde{b}, \bar{b}]$ and $v \in [0, b - \tilde{b})$. Clearly, we have $(b - v)\rho(b) + v = b\rho(b) + (1 - \rho(b))v < b\rho(b) + (1 - \rho(b))(b - \tilde{b}) = \tilde{b}\rho(b) + (b - \tilde{b}) < a(b)$. Hence, $c_{a(b),b,v}$ is type II. Consider $\hat{x}$ again. We have

$$\begin{aligned}
\underline{h}(\hat{x}; a(b), b, v) &= \underline{h}(\tilde{b}\rho(b) + (b - \tilde{b}); a(b), b, v) + \int_{\tilde{b}\rho(b)+(b-\tilde{b})}^{\hat{x}} -\frac{(b - v)\rho(b)}{t - v}\mathrm{d}t \\
&> \underline{h}(\tilde{b}\rho(b) + (b - \tilde{b}); a(b), b, b - \tilde{b}) + \int_{\tilde{b}\rho(b)+(b-\tilde{b})}^{\hat{x}} -\frac{(b - (b - \tilde{b}))\rho(b)}{t - (b - \tilde{b})}\mathrm{d}t \\
&= \underline{h}(\hat{x}; a(b), b, b - \tilde{b}) > c_F(\hat{x}),
\end{aligned}$$

proving that $c_{a(b),b,v}$ is not feasible for any $v \in [0, b - \tilde{b})$.

Finally, for the uniqueness, we only need to show that $c_{a,\tilde{b},0}$ is not feasible for any $a > a(\tilde{b})$. This can be done by a similar argument as in the proof of Claim 7. □

References

An, Mark Yuying, "Logconcavity versus Logconvexity: A Complete Characterization," *Journal of Economic Theory*, 1998, *80* (2), 350–369.

Anderson, Simon P. and Régis Renault, "Pricing, Product Diversity, and Search Costs: A Bertrand-Chamberlin-Diamond Model," *The RAND Journal of Economics*, 1999, pp. 719–735.

_ **and** _ , "Advertising Content," *American Economic Review*, 2006, *96* (1), 93–113.

Armstrong, Mark, "Ordered Consumer Search," *Journal of the European Economic Association*, 2017, *15* (5), 989–1024.

_ **and Jidong Zhou**, "Consumer information and the limits to competition," *American Economic Review*, 2022, *112* (2), 534–77.

_ **, John Vickers, and Jidong Zhou**, "Prominence and Consumer Search," *The RAND Journal of Economics*, 2009, *40* (2), 209–233.

Au, Pak Hung and Keiichi Kawai, "Competitive Information Disclosure by Multiple Senders," *Games and Economic Behavior*, 2020, *119*, 56–78.

_ **and Mark Whitmeyer**, "Attraction versus Persuasion," *arXiv e-prints*, 2018, pp. arXiv–1802.

Bagnoli, Mark and Ted Bergstrom, "Log-concave Probability and Its Applications," *Economic Theory*, 2005, *26* (2), 445–469.

Bar-Isaac, Heski, Guillermo Caruana, and Vicente Cuñat, "Search, Design, and Market Structure," *American Economic Review*, 2012, *102* (2), 1140–60.

Blackwell, David, "Comparison of Experiments," in Jerzy Neyman, ed., *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, University of California Press, Berkeley, California, 1951, pp. 93–102.

Board, Simon and Jay Lu, "Competitive Information Disclosure in Search Markets," *Journal of Political Economy*, 2018, *126* (5), 1965–2010.

Choi, Michael, Anovia Yifan Dai, and Kyungmin Kim, "Consumer Search and Price Competition," *Econometrica*, 2018, *86* (4), 1257–1281.

_ **, Kyungmin Kim, and Marilyn Pease**, "Optimal Information Design for Search Goods," in "AEA Papers and Proceedings," Vol. 109 2019, pp. 550–56.

Condorelli, Daniele and Balázs Szentes, "Information design in the holdup problem," *Journal of Political Economy*, 2020, *128* (2), 681–709.

Dogan, Mustafa and Ju Hu, "Consumer search and optimal information," *The RAND Journal of Economics*, 2022, *53*, 386–403.

Dworczak, Piotr and Giorgio Martini, "The Simple Economics of Optimal Persuasion," *Journal of Political Economy*, 2019, *127* (5), 1993–2048.

Eliaz, Kfir and Ran Spiegler, "A Simple Model of Search Engine Pricing," *The Economic Journal*, 2011, *121* (556).

Gentzkow, Matthew and Emir Kamenica, "A Rothschild-Stiglitz Approach to Bayesian Persuasion," *American Economic Review: Papers & Proceedings*, 2016, *106* (5), 597–601.

Hwang, Ilwoo, Kyungmin Kim, and Raphael Boleslavsky, "Competitive Advertising and Pricing," working paper 2019.

Johnson, Justin P. and David P. Myatt, "On the Simple Economics of Advertising, Marketing, and Product Design," *American Economic Review*, 2006, *96* (3), 756–784.

Kamenica, Emir and Matthew Gentzkow, "Bayesian Persuasion," *American Economic Review*, 2011, *101* (6), 2590–2615.

Kolotilin, Anton, "Optimal Information Disclosure: A Linear Programming Approach," *Theoretical Economics*, 2018, *13* (2), 607–635.

Parthasarathy, Kalyanapuram Rangachari, *Probability Measures on Metric Spaces*, Vol. 352, American Mathematical Society, 2005.

Rayo, Luis and Ilya Segal, "Optimal Information Disclosure," *Journal of Political Economy*, 2010, *118* (5), 949–987.

Rockafellar, R. Tyrrell, *Convex Analysis*, Vol. 18, Princeton University Press, 1970.

Roesler, Anne-Katrin and Balázs Szentes, "Buyer-Optimal Learning and Monopoly Pricing," *American Economic Review*, 2017, *107* (7), 2072–2080.

Wolinsky, Asher, "True Monopolistic Competition as A Result of Imperfect Information," *Quarterly Journal of Economics*, 1986, *101* (3), 493–511.

Zhou, Jidong, "Ordered Search in Differentiated Markets," *International Journal of Industrial Organization*, 2011, *29* (2), 253–262.

Online Appendix

Ju Hu

Dec 14, 2022

Online Appendix A Achievable equilibria and welfare

Based on Proposition 1, we can characterize the set of all equilibria that can arise under any arbitrary feasible signal distribution using conditional unit-elastic demand signal distributions. In this section, we provide some properties about this set. These results can potentially be used to explore a wide range of welfare objectives in this search market, such as the weighted average of the consumer and industry surplus.

Claim A.1. *The set of all achievable equilibria is compact.*

Proof. It is routine to check that

$$K \equiv \big\{(b, v) \bigm| b \in [\mu - s,\, \bar{b}],\, v \in [0, b),\ \text{and}\ c_{a(b),b,v}\ \text{is feasible}\big\}$$

is compact. We omit the details. By Lemma 4, we know the set of all achievable equilibria is compact. □

The next claim shows the interval structure of achievable total welfare and surplus division.

Claim A.2. *(i) The set of all achievable total welfare is an interval $[\mu - s,\, \hat{b}]$ for some $\hat{b} \in [\mu - s,\, \bar{b}]$.*

(ii) For each achievable $b \in [\mu - s,\, \hat{b}]$, there exist $\underline{v} < \bar{v}$ such that (b, v) is an equilibrium if and only if $v \in [\underline{v}, \bar{v}]$.

Proof. Part (i): We show that if $c_{a,b,v}$ with $b > \mu - s$ is feasible then for all $b' \in [\mu - s,\, b)$, there exists v' such that $c_{a,b',v'}$ is also feasible. Fix $b' \in [\mu - s,\, b)$. If $a \leq v$, then Claim A.2 in the online appendix of Dogan and Hu (2022) shows that $c_{a,b',a}$ is feasible. Assume $v < a$. Note $c_{a,b',v}$ is well-defined, i.e., $a \in [0,\, \mu - s)$, $b' \in [\mu - s,\, \bar{b}]$ and $v \in [0, b')$. We proceed to show that $c_{a,b',v}$ is feasible. Let $\pi = \frac{\mu-s-a}{b-a}(b - v)$ and $\pi' = \frac{\mu-s-a}{b'-a}(b' - v)$. Because $v < a$, we have $\pi' > \pi$. For all $x \in [b, 1]$, we have

$$\bar{h}(x; a, b', v) = \bar{h}(b; a, b', v) - \int_b^x \frac{\pi'}{\tilde{x} - v}\mathrm{d}\tilde{x} < \bar{h}(b; a, b, v) - \int_b^x \frac{\pi}{\tilde{x} - v}\mathrm{d}\tilde{x} = \bar{h}(x; a, b, v),$$

where the inequality comes from $\bar{h}(b; a, b', v) < \bar{h}(b'; a, b', v) = s$ and $\pi' > \pi$. Let $\ell(x) = s - \frac{\mu-s-a}{b-a}(x-b)$ and $\ell'(x) = s - \frac{\mu-s-a}{b'-a}(x-b')$. We know $\ell'(x) \leq \ell(x)$ for all $x \geq a$. Therefore, if $c_{a,b',v}$ is type I, it is feasible regardless of whether $c_{a,b,v}$ is type I or II. If $c_{a,b',v}$ is type II, we know $c_{a,b,v}$ is type II too. Moreover, for all $x \in [v+\pi', b'] \subset [v+\pi, b]$, we have

$$\begin{aligned}&\underline{h}(x; a, b', v) = \underline{h}(v+\pi'; a, b', v) - \int_{v+\pi'}^{x} \frac{\pi'}{\tilde{x}-v}\mathrm{d}\tilde{x} \\ <&\underline{h}(v+\pi'; a, b, v) - \int_{v+\pi'}^{x} \frac{\pi}{\tilde{x}-v}\mathrm{d}\tilde{x} = \underline{h}(x; a, b, v),\end{aligned}$$

where the inequality comes from (i) $\underline{h}(v+\pi'; a, b', v) < \underline{h}(v+\pi'; a, b, v)$ and (ii) $\pi' > \pi$. Therefore, in this case, $c_{a,b',v}$ is also feasible.

Part (ii): Suppose c_{a_1,b,v_1} and c_{a_2,b,v_2} are feasible. Assume $v_1 < v_2$. We show that $c_{a(b),b,v}$ is feasible for every $v \in [v_1, v_2]$. By Lemma 4, we know that both $c_{a(b),b,v_1}$ and $c_{a(b),b,v_2}$ are feasible. Because $v \leq v_2$, it is easy to see $\bar{h}(x; a(b), b, v) \leq \bar{h}(x; a(b), b, v_2) \leq c_F(x)$ for all $x \in [b, 1]$. If $c_{a(b),b,v}$ is of type I, it is then feasible. Suppose it is of type II. Because $v + \rho(b)(b-v) \geq v_1 + \rho(b)(b-v_1)$, we know $c_{a(b),b,v_1}$ is of type II too. We have, for all $x \in [v+\rho(b)(b-v), b]$,

$$\begin{aligned}&\underline{h}(x; a(b), b, v) = \underline{h}(v+\rho(b)(b-v); a(b), b, v) - \int_{v+\rho(b)(b-v)}^{x} \frac{\rho(b)(b-v)}{\tilde{x}-v}\mathrm{d}\tilde{x} \\ \leq&\underline{h}(v+\rho(b)(b-v); a(b), b, v_1) - \int_{v+\rho(b)(b-v)}^{x} \frac{\rho(b)(b-v_1)}{\tilde{x}-v_1}\mathrm{d}\tilde{x} = \underline{h}(x; a(b), b, v_1),\end{aligned}$$

where the inequality comes from (i) $\underline{h}(v+\rho(b)(b-v); a(b), b, v) \leq \underline{h}(v+\rho(b)(b-v); a(b), b, v_1)$ and (ii) $\frac{b-v}{\tilde{x}-v} \geq \frac{b-v_1}{\tilde{x}-v_1}$ since $v \geq v_1$ and $\tilde{x} \leq b$. Therefore, $c_{a(b),b,v}$ is also feasible in this case. □

The last claim is about comparative statics of the highest achievable total welfare with respect to the value distribution and search cost.

Claim A.3. *For fixed s, $\hat{b}$ is increasing in F with respect to the mean preserving spread order. For fixed F, $\hat{b}$ is strictly decreasing in search cost s.*

Proof. For the first statement, simply observe that $c_F \subset c_{F'}$ if F' is a mean preserving spread of F. The second statement is proved in the proof of Proposition 2. □

Figure A.1 illustrates the welfare limit for different search costs when the value distribution is uniform. In each panel, the horizontal axis represents the consumer surplus, while the vertical axis represents the industry surplus. Panels (a) - (c)

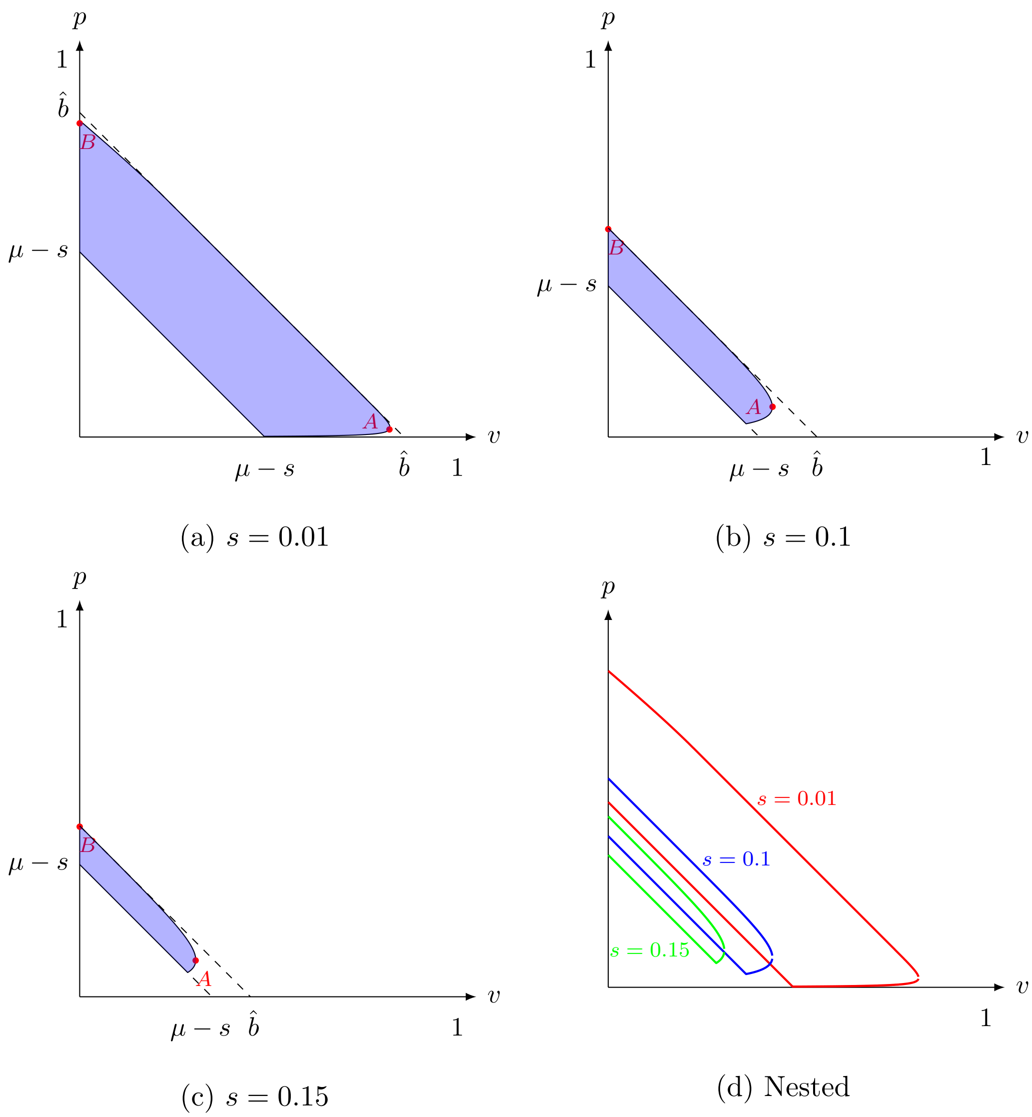


Figure A.1: The welfare limit of uniform value distribution illustrate the welfare limit for $s = 0.01$, $s = 0.1$ and $s = 0.15$ respectively. In each of these panels, each point in the blue area represents a combination of consumer and industry surplus that can be achieved by some signal distribution. Point A represents the one that yields the highest consumer surplus, while B is the one that yields the highest industry surplus. The two dashed lines are downward-sloping-45-degree lines. The lower one indicates the lowest possible total welfare, i.e., $v+p = \mu-s$. The higher one indicates the highest possible total welfare, i.e., $v+p = \hat{b}$. Note that, for all three search costs, the industry extracts all the equilibrium surplus under the industry-optimal signal distribution, leaving the consumers zero surplus, as Propositions 3 and 4 assert. But when $s = 0.01$, the industry-optimal signal distribution does not achieve the highest possible total welfare, as Proposition 4 claims. Panel (d) nests the first three panels. It becomes clear that the achievable welfare moves toward the origin as the search cost increases. This is because both the lower bound $\mu - s$ and the upper bound $\hat{b}$ decrease in search cost, as Claim A.3 states.

Online Appendix B Supplemental materials for value distributions with increasing hazard rate

In this section, we provide three supplemental results when the value distribution has a continuous and positive density with increasing hazard rate.

Claim B.1. *We have $\bar{s} > \hat{s}$.*

Proof. We add s to $\bar{h}$ as an argument and write $\bar{h}(x; a, b, v, s)$ for clarity. Let $\bar{b}$ be such that $\bar{b} = \frac{1-F(\bar{b})}{f(\bar{b})}$. By definition, $c_F(\bar{b}) = \hat{s}$, and $c^{\hat{s}}_{a(\bar{b}),\bar{b},0} \leq c_F$ by Proposition 1. For $x \in [\bar{b}, 1]$, we have

$$\bar{h}(x; 0, \mu - \hat{s}, 0, \hat{s}) = \bar{h}(\bar{b}; 0, \mu - \hat{s}, 0, \hat{s}) - \int_{\bar{b}}^{x} \frac{\mu - \hat{s}}{\tilde{x}} \mathrm{d}\tilde{x} < \hat{s} - \int_{\bar{b}}^{x} \frac{\bar{b}(1 - F(\bar{b}))}{\tilde{x}} \mathrm{d}\tilde{x}$$
$$= \bar{h}(\bar{b}; a(\bar{b}), \bar{b}, 0, \hat{s}) - \int_{\bar{b}}^{x} \frac{\bar{b}(1 - F(\bar{b}))}{\tilde{x}} \mathrm{d}\tilde{x} = \bar{h}(x; a(\bar{b}), \bar{b}, 0, \hat{s}) \leq c_F(x),$$

where the strict inequality comes from (i) $\bar{h}(\mu-\hat{s}; 0, \mu-\hat{s}, 0, \hat{s}) = \hat{s}$ and $\bar{h}$ is decreasing in x and (ii) $\mu-\hat{s} = c_F(0)-c_F(\bar{b}) = \int_0^{\bar{b}}(1-F(\tilde{x}))\mathrm{d}\tilde{x} > \bar{b}(1-F(\bar{b}))$. For $x \in [\mu-\hat{s},\ \bar{b})$, we have $\bar{h}(x; 0, \mu - \hat{s}, 0, \hat{s}) \leq \hat{s} < c_F(x)$. Thus, by uniform continuity, there exists $s' > \hat{s}$ such that for all $s \in [\hat{s}, s']$, we have $\bar{h}(x; 0, \mu - s, 0, s) \leq c_F(x)$ for all $x \in [\mu - \hat{s},\ 1]$.

Define $\kappa(x,s) : [\mu - s',\, \mu - \hat{s}] \times [\hat{s}, s'] \to \mathbb{R}$ as

$$\kappa(x,s) = \begin{cases} \mu - x, & \text{if } x \in [\mu - s',\, \mu - s], \\ \bar{h}(x; 0, \mu - s, 0, s), & \text{if } x \in [\mu - s,\, \mu - \hat{s}]. \end{cases}$$

Because f is positive, we know $\kappa(x, \hat{s}) = \mu - x < c_F(x)$ for all $x \in [\mu - s',\, \mu - \hat{s}]$. By uniform continuity of κ, there exists $s'' \in (\hat{s}, s']$ such that for all $s \in (\hat{s}, s'')$, $\kappa(x, s'') \leq c_F(x)$ for all $x \in [\mu - s',\, \mu - \hat{s}]$. This implies that for all $s \in (\hat{s}, s'')$, $\underline{h}(x; 0, \mu - s, 0, s) \leq c_F(x)$ for all $x \in [\mu - s,\, \mu - \hat{s}]$. Therefore, for all $s \in (\hat{s}, s'')$, $\underline{h}(x; 0, \mu - s, 0, s) \leq c_F(x)$ for all $x \in [\mu - s,\, 1]$. This implies that $c^s_{0,\mu-s,0}$ is feasible if $s \in (\hat{s}, s'')$, proving $\bar{s} > \hat{s}$. □

Claim B.2. *If $s \in (\hat{s}, \bar{s}]$, then $\hat{b} < \bar{b}$.*

Proof. By Lemma 4, it suffices to show that there is no $v \in [0, \bar{b})$ such that $c_{a(\bar{b}),\bar{b},v}$ is feasible. Let $\bar{b}'$ be such that $\bar{b}' = \frac{1-F(\bar{b}')}{f(\bar{b}')}$. By definition, $c_F(\bar{b}') = \hat{s}$. Because $s > \hat{s}$, we have $\bar{b} < \bar{b}'$. By increasing hazard rate, we have $\bar{b}(1 - F(\bar{b})) < x(1 - F(x))$ for all $x \in (\bar{b}, \bar{b}']$. Consider $\bar{h}(x; a(\bar{b}), \bar{b}, 0)$. For any $x \in (\bar{b}, \bar{b}']$,

$$\bar{h}(x; a(\bar{b}), \bar{b}, 0) = s - \int_{\bar{b}}^{x} \frac{\bar{b}(1 - F(\bar{b}))}{\tilde{x}} \mathrm{d}\tilde{x} > s - \int_{\bar{b}}^{x} (1 - F(\tilde{x}) \mathrm{d}\tilde{x} = c_F(x),$$

implying that $c_{a(\bar{b}),\bar{b},0}$ is not feasible. For any $v \in (0, \bar{b})$, we then have $\bar{h}(x; a(\bar{b}), \bar{b}, v) > \bar{h}(x; a(\bar{b}), \bar{b}, 0)$ for all $x > \bar{b}$, implying $c_{a(\bar{b}),\bar{b},v}$ is not feasible either. □

Claim B.3. *Assume f is increasing. Let $\tilde{b}_s$ be the optimal industry surplus (total welfare) when the search cost is s. Then, $\lim_{s \downarrow 0} \tilde{b}_s < 1$.*

Proof. We add search cost s as an explicit argument to $a(b)$ and $\rho(b)$, and write $a(b,s)$ and $\rho(b,s)$ respectively. Let $\pi(b,s) = \rho(b,s)b$. Define

$$\underline{h}^{\dagger}(x; b, s) \equiv \mu - \pi(b,s) - \pi(b,s) \log \frac{x}{\pi(b,s)},\ \forall x \in [\pi(b,s),\, b].$$

Note that $\underline{h}^{\dagger}(x; b, s)$ is just a short notation for $\underline{h}(x; a(b,s), b, 0, s)$. Let $B \equiv \{b \in [\mu, 1] \mid \underline{h}^{\dagger}(x; b, 0) \leq c_F(x),\ \forall x \in [\pi(b, 0), b]\}$. Because $\pi(\mu, 0) = \mu$, we know $\mu \in B \neq \emptyset$. Let $b^{\dagger} \equiv \sup B$. We must have $b^{\dagger} < 1$. To see this, suppose by contradiction that there exists $\{b_n\}_n \subset B$ such that $\lim_n b_n = 1$. Fix an arbitrary x close 1. We have $\lim_n \underline{h}^{\dagger}(x; b_n, 0) = \mu > c_F(x)$, where the equality comes from the fact that $\pi(1, 0) = 0$. This implies that $\underline{h}^{\dagger}(x; b_n, 0) > c_F(x)$, contradicting the construction of B. Therefore, we must have $b^{\dagger} < 1$.

Using a similar argument as that in Claim 10, we can also show that there must exist $x^\dagger \in (\pi(b^\dagger,0), b^\dagger)$ such that $\underline{h}^\dagger(x^\dagger; b^\dagger, 0) = c_F(x^\dagger)$. Pick $s^\dagger \in (0, \tilde{s})$ such that $b^\dagger - \frac{s^\dagger}{\rho(b^\dagger,0)} > x^+$, where, recall, $\tilde{s}$ is defined in Proposition 3. We proceed to show that $\tilde{b}_s < b^\dagger - \frac{s}{\rho(b^\dagger,0)}$ for all $s \in (0, s^\dagger)$, which will prove the desired result. Suppose, by contradiction, that $\tilde{b}_s \geq b^\dagger - \frac{s}{\rho(b^\dagger,0)}$ for some $s \in (0, s^\dagger)$. We have $\pi(\tilde{b}_s, s) \leq \pi(b^\dagger - \frac{s}{\rho(b^\dagger,0)}, s) = \rho(b^\dagger - \frac{s}{\rho(b^\dagger,0)}, s)(b^\dagger - \frac{s}{\rho(b^\dagger,0)}) = \rho(b^\dagger,0)(b^\dagger - \frac{s}{\rho(b^\dagger,0)}) < \pi(b^\dagger, 0)$, where the first inequality comes from Claim 1. The second equality comes from $\rho(b^\dagger - \frac{s}{\rho(b^\dagger,0)}, s) = \rho(b^\dagger, 0)$. These inequalities imply that $\hat{x} \in [\pi(\tilde{b}_s, s), \tilde{b}_s]$. Then $\underline{h}^\dagger(x^\dagger; \tilde{b}_s, s) = \mu - \pi(\tilde{b}_s, s) - \pi(\tilde{b}_s, s) \log \frac{x}{\pi(\tilde{b}_s, s)} > \mu - \pi(b^\dagger, 0) - \pi(b^\dagger, 0) \log \frac{x}{\pi(b^\dagger, 0)} = \underline{h}^\dagger(x^\dagger; b^\dagger, 0) = c_F(x^\dagger)$. This implies that $c^s_{a(\tilde{b}_s), \tilde{b}_s, s}$ is not feasible, as Claim 10 has shown that $c^s_{a(\tilde{b}_s), \tilde{b}_s, 0}$ is of type II. This contradicts the fact that $c^s_{a(\tilde{b}_s), \tilde{b}_s, 0}$ is the industry-optimal signal distribution. Therefore, we must have $\tilde{b}_s < b^\dagger - \frac{s}{\rho(b^\dagger,0)}$ for all $s \in (0, s^\dagger)$ implying $\lim_{s \downarrow 0} \tilde{b}_s \leq b^\dagger < 1$. □

Online Appendix C Proof of Proposition 5

Denote the signal distribution that induces (σ, v) by G. For notational simplicity, we write c, instead of c_G, as the incremental benefit function of G. Let $\pi = -(b - v)c'(b-)$ for some (any) $b \in \operatorname{supp}\sigma$ be the expected profit of a matched firm under (σ, v). Because of the firms' indifference conditions implied by (10), π is independent of the choice of b. Note that, the equilibrium industry surplus can then be expressed as

$$b_e - v = \frac{\int(-c'(b-))b\mathrm{d}\sigma(b)}{\int(-c'(b))\mathrm{d}\sigma(b)} - v = \frac{\pi}{\int(-c'(b))\mathrm{d}\sigma(b)}. \tag{C.1}$$

Moreover, throughout the proof, we continue to add search cost s as a parameter to $c_{a,b,v}$, $\underline{h}$, and $\bar{h}$ for clarity, as we did in the proof of Proposition 2.

The whole proof of Proposition 5 is involved. Section C.1 below provides two preliminary results that will be used in the later proof. Section C.2 contains the main proof.

C.1 Preliminaries

Claim C.1 summarizes some simple observations about c function, which will guarantee that some later constructions are well-defined. Claim C.2 itself can be considered as another proof of Proposition 2. It provides a stronger result than the one obtained in the proof of Proposition 2.[24]

[24]The proof of Proposition 2 given in the appendix is simpler and easier to interpret.

Claim C.1. *At any x, $c(x) > 0$ implies $c'(x-) < 0$. As a result, we have (i) $\pi > 0$; (ii) $c'(b-)$ is negative and strictly increasing over* suppσ*; and (iii) $c(b)$ is strictly decreasing over* suppσ*.*

Proof. Since c is decreasing, $c'(x-) \leq 0$ and is increasing. If $c(x) > 0$ and $c'(x-) = 0$, we have $c'(x'-) = 0$ for all $x' \in [x, 1]$. Then $c(1) = c(x) > 0$, a contradiction. Hence, $c(x) > 0$ implies $c'(x-) < 0$.

It is clear that $\pi \geq 0$. Suppose, by contradiction, $\pi = 0$. Pick any $b \in \text{supp}\sigma$ so that $c(b) > 0$. Such b must exist because of (9) and $s > 0$. Then we must have $b = v$ since $(b - v)(-c'(b-)) = \pi = 0$. Pick any $x > b$ such that $c(x) > 0$. We have $(x - v)(-c'(x-)) > 0 = \pi$, contradicting (10). Therefore, $\pi > 0$. This directly implies that $c'(b-) \neq 0$ for all $b \in \text{supp}\sigma$, or equivalently $c'(b-) < 0$ for all $b \in \text{supp}\sigma$. It also implies that $b > v$ for all $b \in \text{supp}\sigma$. Since $c'(b-) = -\frac{\pi}{b-v}$, it is clear that $c'(b-)$ is strictly increasing over suppσ. Finally, if $b_1, b_2 \in \text{supp}\sigma$ and $c(b_1) = c(b_2)$, we know $c'(b_2-) = 0$, contradicting the above analysis. Hence $c(b)$ is strictly decreasing over suppσ. □

Claim C.2. *Suppose $0 < s' < s \leq \bar{s}$ and $c^s_{a,b,v}$ is feasible. Let b' be the unique solution to $\bar{h}(b'; a, b, v, s) = s'$. Then, $c^{s'}_{a',b',v}$ is feasible for some a'.*

Proof. We clearly have $b' > b$. Let $\ell'(x)$ be the contingent line of $c^s_{a,b,v}$ at b', i.e., $\ell'(x) = -\frac{\pi}{b'-v}(x - v) + s'$, where π is the expected profits of a matched firm under $c^s_{a,b,v}$. Denote by a' the intersection of ℓ' and the downward-sloping-45-degree line of c_{F_0}. See Figure C.1 for an illustration of a', b' and ℓ'. Note that we have $a < a'$.

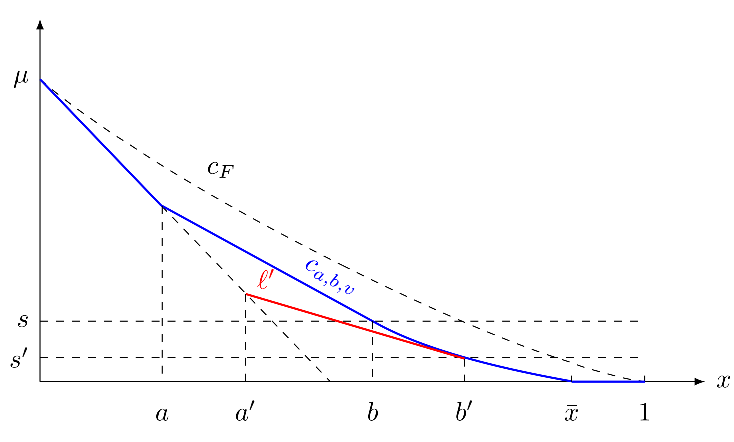


Figure C.1: Proof of Claim C.2

By construction, the equilibrium probability of trade under $c^{s'}_{a',b',v}$ is $-c^s_{a,b,v}(b'-) = \frac{\pi}{b'-v}$, and thus the corresponding expected profit of a matched firm is $\frac{\pi}{b'-v}(b' - v) = \pi$, which is the same as that under $c^s_{a,b,v}$. This directly implies that $c^{s'}_{a',b',v}$ coincides with $c^s_{a,b,v}$ over $[b',1]$, because for any $x \in [b',1]$,

$$\begin{aligned}\bar{h}(x;a',b',v,s') &= s' - \pi \log \frac{x-v}{b'-v} = \bar{h}(b';a,b,v,s) - \pi \log \frac{x-v}{b'-v} \\ &= s - \pi \log \frac{b'-v}{b-v} - \pi \log \frac{x-v}{b'-v} = s - \pi \log \frac{x-v}{b-v} = \bar{h}(x;a,b,v,s).\end{aligned}$$

Hence, $c^{s'}_{a',b',v} \le c_F$ over $[b',1]$.

Consider the interval $[0,b']$. If $a' \le v+\pi$, $c^{s'}_{a',b',v}$ is type I and we immediately know that $c^{s'}_{a',b',v} \le c_F$ over this interval, because $\ell' \le c^s_{a,b,v} \le c_F$, where the first inequality comes from the fact that ℓ' by construction is tangent to $c^s_{a,b,v}$. Suppose $a' > v+\pi$, i.e., $c^{s'}_{a',b',v}$ is type II. In what follows, we proceed to verify that $\max\{\underline{h}(x;a',b',v,s'), \ell'(x)\} \le c_F(x)$ for $x \in [v+\pi, b']$ and $\underline{h}(b';a',b',v,s') \le s'$. Note that, by construction, we have

$$\underline{h}(x;a',b',v,s') = \mu - v - \pi - \pi \log \frac{x-v}{\pi} = \underline{h}(x;a,b,v,s), \ \forall x \ge v+\pi.$$

Hence, we need to show $\max\{\underline{h}(x;a,b,v,s), \ell'(x)\} \le c_F(x)$ for $x \in [v+\pi, b']$ and $\underline{h}(b';a,b,v,s) \le s'$.

Consider first the case $a > v+\pi$, i.e., $c^s_{a,b,v}$ is type II too. Because $c^s_{a,b,v}$ is feasible, we know $\max\{\underline{h}(x;a,b,v,s), \ell(x)\} \le c_F(x)$ for all $x \in [v+\pi, b]$, where $\ell(x) \equiv \frac{\pi}{b-v}(x-b)+s$ as in the construction of $c^s_{a,b,v}$, and $\underline{h}(b;a,b,v,s) \le s$. Since $\ell'(x) \le \ell(x)$ when $x \le b$ (see Figure C.1), we know $\max\{\underline{h}(x;a,b,v,s), \ell'(x)\} \le c_F(x)$ for all $x \in [v+\pi, b]$. Because $\underline{h}(x;a,b,v,s)$ and $\bar{h}(x;a,b,v,s)$ have the same slope $-\frac{\pi}{x-v}$ when $x \ge b$, $\underline{h}(b;a,b,v,s) \le s = \bar{h}(b;a,b,v,s)$ implies that $\underline{h}(x;a,b,v,s) \le \bar{h}(x;a,b,v,s)$ for $x \ge b$. This observation implies that $\max\{\underline{h}(x;a,b,v,s), \ell'(x)\} \le \bar{h}(x;a,b,v,s) \le c_F(x)$ for all $x \in [b,b']$, and $\underline{h}(b';a,b,v,s) \le \bar{h}(b',a,b,v,s) = s'$.

Consider next the case $a \le v+\pi < a'$. In this case, $c^s_{a,b,v}$ is type I, but function $\underline{h}(x;a,b,v,s)$ is still well-defined. Clearly, we have $\ell(v+\pi) > \mu - v - \pi = \underline{h}(v+\pi;a,b,v,s)$. Moreover, for any $x \in [v+\pi, b]$, $\frac{\partial \underline{h}(x;a,b,v,s)}{\partial x} = -\frac{\pi}{x-v} < -\frac{\pi}{b-v}$. Hence, for any $x \in [v+\pi, b]$,

$$\underline{h}(x;a,b,v,s) = \underline{h}(v+\pi;a,b,v,s) + \int_{v+\pi}^{x} (-\frac{\pi}{t-v}) \mathrm{d}t < \ell(v+\pi) + \int_{v+\pi}^{x} (-\frac{\pi}{b-v}) \mathrm{d}t = \ell(x).$$

Thus, we have $\max\{\underline{h}(x;a,b,v,s), \ell'(x)\} \le \ell(x) \le c_F(x)$ for $x \in [v+\pi, b]$. Moreover, evaluating the above inequality at $x = b$ yields $\underline{h}(b;a,b,v,s) \le \ell(b) = s$. Then, we can apply the same argument as in the previous case to show that $\max\{\underline{h}(x;a,b,v,s), \ell'(x)\} \le c_F(x)$ for $x \in [b,b']$ and $\underline{h}(b';a,b,v,s) \le s'$.

In summary, we have shown, for all possible cases, that $c^{s'}_{a',b',v} \le c_F$. Therefore, $c^{s'}_{a',b',v}$ is feasible and yields industry surplus $b' - v > b - v$. □

C.2 Proof of Proposition 5

Proposition 5 is first proved for the case where the support of strategy σ contains only two signal cutoffs (Claim C.3). It is then extended to the cases where the support of σ contains finitely many or infinitely many signal cutoffs (Claims C.4 and C.5, respectively).

Claim C.3. *If* suppσ *contains only two signal cutoffs, Proposition 5 holds.*

Proof. Suppose supp$\sigma = \{b_1, b_2\}$ and $b_1 < b_2$. By (9) and Claim C.1, we have $c(b_2) < s < c(b_1)$. With slight abuse of notation, we use $\sigma(b_i)$ to denote the probability of cutoff b_i (or equivalently, the probability of setting price $b_i - v$) under strategy σ.

Because of (10), we know (b_1, v) is a pure strategy equilibrium in the market with search cost $s_1 \equiv c(b_1)$. By Proposition 1, we know $c^{s_1}_{a_1,b_1,v} \leq c$ is feasible, where $a_1 = \mathbb{E}_G(q|q < b_1)$. Let $b^\dagger \in [b_1, b_2]$ be the unique intersection of $\bar{h}(\cdot; a_1, b_1, v, s_1)$ and the horizontal line of value s. That is, $b^\dagger$ satisfies $\bar{h}(b^\dagger; a_1, b_1, v, s_1) = s$. See panel (a) of Figure C.2 for an illustration. By Claim C.2, $c^s_{a^\dagger,b^\dagger,v}$ is feasible for some $a^\dagger$. Therefore, if $b^\dagger \geq b_e$, we immediately obtain the desired result. In the rest of this proof, we assume $b^\dagger < b_e$. For clarity, we divide the whole proof into small steps.

Step 1: Introducing $(\hat{x}, \hat{y})$.

Let ℓ_1 and ℓ_2, respectively, be the left tangent lines of c at b_1 and b_2. That is, for $i = 1, 2$,

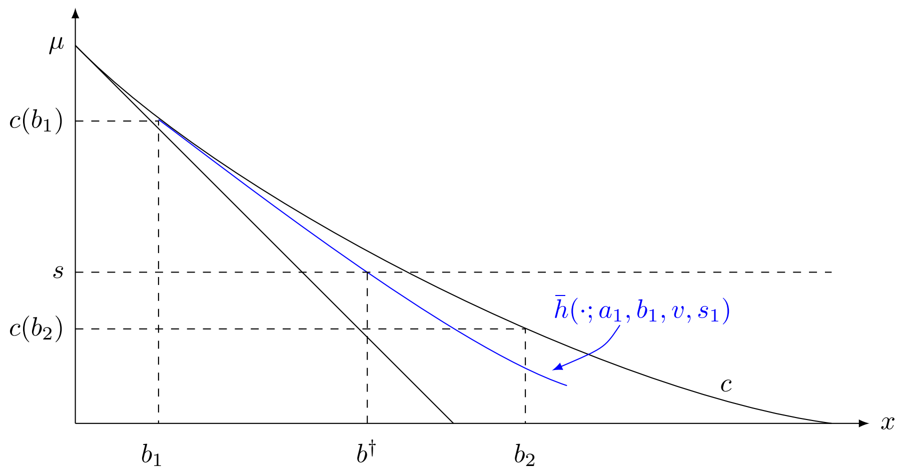
$$\ell_i(x) \equiv c'(b_i-)(x - b_i) + c(b_i).$$

By Claim C.1, ℓ_1 and ℓ_2 have different slopes. Therefore, these two lines have a unique intersection, denoted by $(\hat{x}, \hat{y})$. Because both lines are below c, we have $\ell_1(b_2) \leq c(b_2) = \ell_2(b_2)$ and $\ell_2(b_1) \leq c(b_1) = \ell_1(b_1)$. This implies that $b_1 \leq \hat{x} \leq b_2$ and $c(b_2) \leq \hat{y} \leq c(b_1)$. See panel (b) of Figure C.2 for an illustration of ℓ_1, ℓ_2 and $(\hat{x}, \hat{y})$.

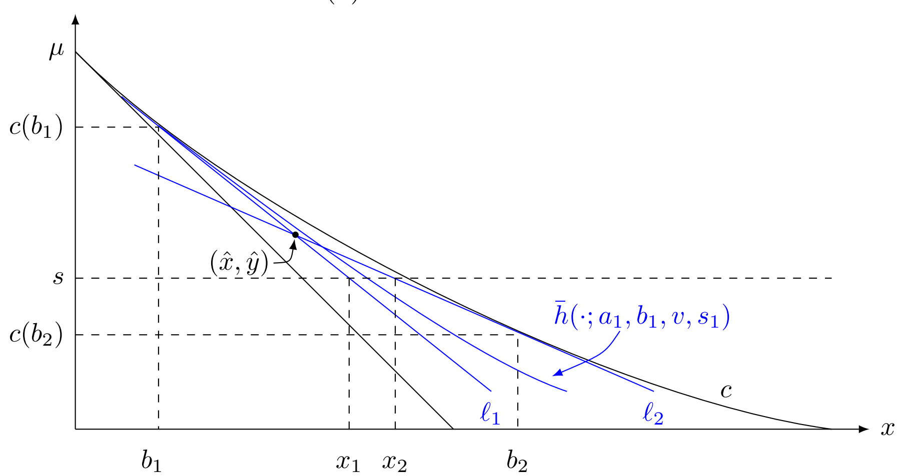
Step 2: Introducing x_1 and x_2.

For $i = 1, 2$, let x_i be the unique intersection of ℓ_i and the horizontal line of value s. Such intersection exists because $c'(b_i-) < 0$ by Claim C.1. See panel (b) of Figure C.2 for an illustration of x_1 and x_2. We can explicitly write $x_i = \frac{s - c(b_i)}{c'(b_i-)} + b_i$. It is easy to see that $x_2 \leq x_1$ if $s \geq \hat{y}$, and $x_1 < x_2$ if $s < \hat{y}$. Note also that we always have $b^\dagger > x_1$ since $\bar{h}(\cdot; a_1, b_1, v, s_1)$ is always above ℓ_1 over $(b_1, b_2]$.

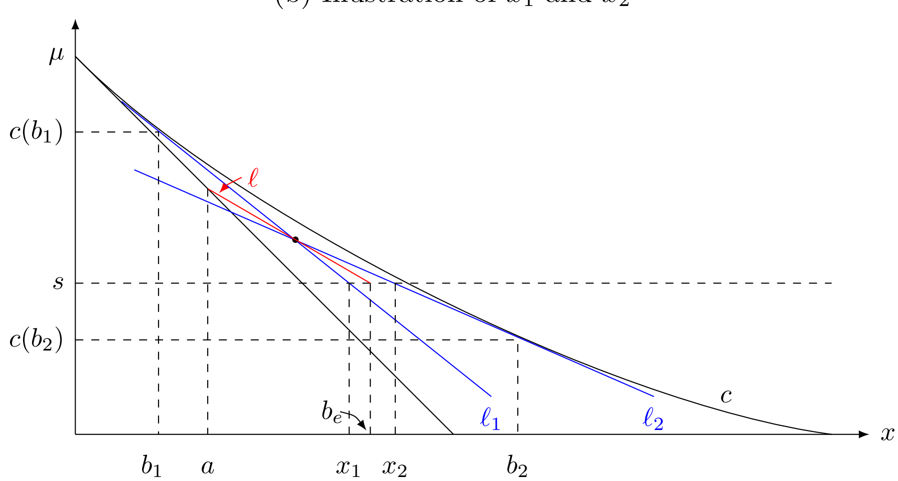
Step 3: $b^\dagger < b_e$ implies $s < \hat{y}$, or equivalently $x_1 < x_2$.



(a) Illustration of $b^{\dagger}$



(b) Illustration of x_1 and x_2



(c) Illustration of ℓ

Figure C.2: Proof of Claim C.3

For notational simplicity, let

$$\beta_i \equiv \frac{\sigma(b_i)(-c'(b_i-))}{\sigma(b_1)(-c'(b_1-)) + \sigma(b_2)(-c'(b_2-))} > 0,\ \ i = 1, 2, \tag{C.2}$$

so that $b_e = \beta_1 b_1 + \beta_2 b_2$ by (11). It is easy to see that

$$\sum_{i=1}^{2} \beta_i x_i = \sum_{i=1}^{2} \beta_i \frac{s - c(b_i)}{c'(b_i-)} + \sum_{i=1}^{2} \beta_i b_i = \sum_{i=1}^{2} \beta_i b_i = b_e,$$

where the second equality comes from (9). This means that b_e is also a convex combination of x_1 and x_2. If $s \geq \hat{y}$, we know $b_e \leq \max\{x_1, x_2\} = x_1 < b^\dagger$, contradicting the assumption that $b^\dagger < b_e$. Therefore, we have $s < \hat{y}$.

Step 4: Constructing $c^s_{a,b_e,v}$.

Let ℓ be the straight line that passes through $(\hat{x}, \hat{y})$ and (b_e, s):

$$\ell(x) = -\frac{\hat{y} - s}{b_e - \hat{x}}(x - b_e) + s.$$

It is the red line in panel (c) of Figure C.2. It intersects the downward-sloping-45-degree line of c_{F_0} at some a. We will show that $c^s_{a,b_e,v}$ is feasible, which will give the desired result. By construction, we have $\ell^{-1}(y) = \beta_1 \ell_1^{-1}(y) + \beta_2 \ell_2^{-1}(y)$. Thus, it is easy to calculate that the probability of trade per match under $c^s_{a,b_e,v}$ is

$$\frac{\hat{y} - s}{b_e - \hat{x}} = \frac{1}{\frac{\beta_1}{-c'(b_1-)} + \frac{\beta_2}{-c'(b_2-)}} = \sum_i (-c'(b_i-))\sigma(b_i). \tag{C.3}$$

Then, the corresponding expected profit of a matched firm is

$$\frac{\hat{y} - s}{b_e - \hat{x}}(b_e - v) = \sum_i (-c'(b_i-))\sigma(b_i) \frac{\pi}{\sum_i (-c'(b_i-))\sigma(b_i)} = \pi, \tag{C.4}$$

the same as that in (σ, v) under c.

Step 5: $c^s_{a,b_e,v} \leq c$ over $[b_e, 1]$.

Consider the interval $[b_e, b_2]$ first. To show that $c^s_{a,b_e,v} \leq c$ over this range, it suffices to show that $\bar{h}(x; a, b_e, v, s) \leq \ell_2(x)$ for all $x \in [b_e, b_2]$. Since $\bar{h}(\cdot; a, b_e, v, s)$ is convex and $\bar{h}(b_e; a, b_e, v, s) = s = \ell_2(x_2) < \ell_2(b_e)$, we only need to show that $\bar{h}(b_2; a, b_e, v, s) \leq \ell_2(b_2) = c(b_2)$. Using the equilibrium condition $\sigma(b_1)c(b_1) + \sigma(b_2)c(b_2) = s$, i.e., (9), we can obtain $\sigma(b_1) = \frac{s - c(b_2)}{c(b_1) - c(b_2)}$ and $\sigma(b_2) = \frac{c(b_1) - s}{c(b_1) - c(b_2)}$. Plugging these expressions into (C.1), we have

$$b_e - v = \frac{\pi(c(b_1) - c(b_2))}{(s - c(b_2))(-c'(b_1-)) + (c(b_1) - s)(-c'(b_2-))} \equiv \theta(s).$$

For $\tilde{s} \in [c(b_2), s]$, define $\phi(\tilde{s}) \equiv \tilde{s} - \pi \log \frac{b_2 - v}{\theta(\tilde{s})}$. To show that $\bar{h}(b_2; a, b_e, v, s) \leq c(b_2)$, it is equivalent to showing that $\phi(s) \leq c(b_2)$.

It is easy to see that $\theta(c(b_2)) = b_2 - v$. Thus, $\phi(c(b_2)) = c(b_2)$. Moreover, it is easy to calculate

$$\begin{aligned}\frac{\mathrm{d}\phi}{\mathrm{d}\tilde{s}} &= 1 + \frac{\pi}{\theta(\tilde{s})} \frac{\mathrm{d}\theta(\tilde{s})}{\mathrm{d}\tilde{s}} \\ &= \frac{\left(\frac{\tilde{s} - c(b_1)}{c'(b_1-)} + b_1\right) - \left(\frac{\tilde{s} - c(b_2)}{c'(b_2-)} + b_2\right)}{(-c'(b_1-))(-c'(b_2-))[(s - c(b_2))(-c'_G(b_1-)) + (c(b_1) - s)(-c'(b_2-))]}.\end{aligned}$$

Because $\tilde{s} \leq s < \hat{y}$, we know $\frac{\tilde{s} - c(b_1)}{c'(b_1-)} + b_1 < \frac{\tilde{s} - c(b_2)}{c'(b_2-)} + b_2$, implying $\frac{\mathrm{d}\phi(\tilde{s})}{\mathrm{d}\tilde{s}} < 0$. Therefore, $\phi(s) < \phi(c(b_2)) = c(b_2)$.

Next, consider the interval $[b_2, 1]$. If $c(b_2) = 0$, the above analysis implies that $\bar{h}(b_2; a, b_e, v, s) \leq 0$, which in turn implies that $c^s_{a,b_e,v} = 0$ over $[b_2, 1]$. Hence $c^s_{a,b_e,v} \leq c$ over this interval. If $c(b_2) > 0$, then (10) implies that (b_2, v) is a pure strategy equilibrium in the market with search cost $s_2 \equiv c(b_2)$. By Proposition 1, we know $c^{s_2}_{a_2,b_2,v} \leq c$ where $a_2 = \mathbb{E}_G(q|q < b_2)$. In particular, $\bar{h}(x; a_2, b_2, v, s_2) \leq c(x)$ for $x \in [b_2, 1]$. Note $\bar{h}(x; a_2, b_2, v, s_2) = s_2 - \pi \log \frac{x-v}{b_2-v}$ while $\bar{h}(x; a, b_e, v, s) = s - \pi \log \frac{x-v}{b_e-v}$. Hence, these two curves (as functions of x) have exactly the same slope $-\frac{\pi}{x-v}$ over $[b_2, 1]$. Because we have shown $\bar{h}(b_2; a, b_e, v, s) \leq c(b_2) = \bar{h}(b_2; a_2, b_2, v, s_2)$, we know $\bar{h}(x; a, b_e, v, s) \leq \bar{h}(b_2; a_2, b_2, v, s_2) \leq c(x)$ for $x \in [b_2, 1]$.

Step 6: $c^s_{a,b_e,v}$ is feasible.

If $a \geq v + \pi$, then $c^s_{a,b_e,v}$ is type I. Because $\ell \leq \max\{\ell_1, \ell_2\} \leq c$, which can be directly seen from panel (c) of Figure C.2, we immediately know that $c^s_{a,b_e,v} \leq c$ over $[0, b_e]$. Because we have shown $c^s_{a,b_e,v} \leq c$ over $[b_e, 1]$ in the previous step, we know $c^s_{a,b_e,v}$ is feasible.

Assume $a < v+\pi$. Then $c^s_{a,b_e,v}$ is type II. We verify that $\max\{\underline{h}(x; a, b_e, v, s), \ell(x)\} \leq c(x)$ for $x \in [v + \pi, b_e]$ and $\underline{h}(b_e; a, b_e, v, s) \leq s$. These inequalities, together with $c^s_{a,b_e,v} \leq c$ over $[b_e, 1]$ from the previous step, will imply the feasibility of $c^s_{a,b_e,v}$.

To show $\max\{\underline{h}(x; a, b_e, v, s), \ell(x)\} \leq c(x)$ for $x \in [v + \pi, b_e]$, it suffices to show $\underline{h}(x; a, b_e, v, s) \leq c(x)$ for $x \in [v + \pi, b_e]$, because we have already argued that $\ell \leq c$. By construction, $\underline{h}(x; a, b_e, v, s) = \mu - v - \pi - \pi \log \frac{x-v}{\pi} = \underline{h}(x; a_1, b_1, v, s_1)$ for $x \geq v + \pi$. If $c^{s_1}_{a_1,b_1,v}$ is type II, we know $\max\{\underline{h}(x; a_1, b_1, v, s_1), \ell_1(x)\} \leq c(x)$ for all $x \in [v + \pi, b_1]$, since $c^{s_1}_{a_1,b_1,v}$ is feasible. This immediately implies $\underline{h}(x; a, b_e, v, s) \leq \max\{\underline{h}(x; a_1, b_1, v, s_1), \ell_1(x)\} \leq c(x)$ for $x \in [v + \pi, b_1]$. If $c^{s_1}_{a_1,b_1,v}$ is type I, we know $\underline{h}(x; a_1, b_1, v, s_1) \leq \ell_1(x)$ for $x \in [v + \pi, b_1]$. Thus, $\underline{h}(x; a_1, b_1, v, s_1) \leq c(x)$ for $x \in [v + \pi, b_1]$. Moreover, in both cases, we have $\underline{h}(b_1; a_1, b_1, v, s_1) \leq s_1 = c(b_1) = \bar{h}(b_1; a_1, b_1, v, s_1)$. Observe also that $\frac{\partial \underline{h}(x; a_1, b_1, v, s_1)}{\partial x} = -\frac{\pi}{x-v} = \frac{\partial \bar{h}(x; a_1, b_1, v, s_1)}{\partial x}$ for $x \geq b_1$. Therefore, $\underline{h}(x; a_1, b_1, v, s_1) \leq \bar{h}(x; a_1, b_1, v, s_1) \leq c(x)$ for $x \in [b_1, b_e]$, where the second inequality comes from the feasibility of $c^{s_1}_{a_1, b_1, v}$. This implies $\underline{h}(x; a, b_e, v, s) \leq c(x)$ for $x \in [b_1, b_e]$. Finally, the inequality $\underline{h}(x; a_1, b_1, v, s_1) \leq \bar{h}(x; a_1, b_1, v, s_1)$ also implies $\underline{h}(b_e; a_1, b_1, v, s_1) \leq \bar{h}(b_e; a_1, b_1, v, s_1) < \bar{h}(b^\dagger; a_1, b_1, v, s_1) = s_1$, where the second inequality comes from the assumption that $b^\dagger < b_e$ and the fact that $\bar{h}$ is strictly decreasing in x. This completes the proof. □

The next claim extends Claim C.3 to finitely supported equilibrium strategy σ.

Claim C.4. *If* suppσ *contains finitely many signal cutoffs, Proposition 5 holds.*

Proof. Suppose supp$\sigma = \{b_1, \ldots, b_n\}$. Consider the following optimization problem

$$u \equiv \min_{\tilde{\sigma} \text{ mixes over supp}\sigma} \sum_i \tilde{\sigma}(b_i)(-c'(b_i-))$$
$$\text{s.t.} \sum_i \tilde{\sigma}(b_i) c(b_i) = s.$$

By Corollary 17.1.5 in Rockafellar (1970), it has a solution $\hat{\sigma}$ whose support contains at most two b_i's. If supp$\hat{\sigma} = \{b_i\}$ for some i, we know $c(b_i) = s$. By (10), (b_i, v) is a pure strategy equilibrium. Thus, $c^s_{a, b_i, v}$ is feasible by Proposition 1, where $a = \mathbb{E}_G[q | q < b_i]$. Moreover,

$$b_i - v = \frac{\pi}{-c'(b_i-)} = \frac{\pi}{u} \geq \frac{\pi}{\sum_i \sigma(b_i)(-c'(b_i-))} = b_e - v,$$

where the inequality is because σ is feasible to the above minimization problem. This implies that $c^s_{a, b_i, v}$ is the desired one.

Suppose supp$\hat{\sigma} = \{b_i, b_j\}$ for some $i \neq j$. Similarly as above, $(\hat{\sigma}, v)$ is a mixed strategy equilibrium. Let $\hat{b}_e$ be the expected total welfare of this equilibrium. We have

$$\hat{b}_e - v = \frac{\pi}{u} \geq \frac{\pi}{\sum_i \sigma(b_i)(-c'(b_i-))} = b_e - v.$$

By Claim C.3, we know that there exist a and $b' \geq \hat{b}_e$ such that $c^s_{a, b', v}$ is feasible. Since $b' \geq \hat{b}_e \geq b_e$, we obtain the desired result. □

The last claim further extends Claims C.3 and C.4 to infinitely supported strategy σ, which completes the proof of Proposition 5.

Claim C.5. *If* suppσ *contains infinitely many signal cutoffs, Proposition 5 holds.*

Proof. By Theorem 6.3 in Parthasarathy (2005), we can find a sequence of mixed strategies $\{\sigma_n\}$ such that (i) $\text{supp}\sigma_n$ is finite and $\text{supp}\sigma_n \subset \text{supp}\sigma$, and (ii) σ_n converges weakly to σ. Thus, we have[25]

$$\lim_n \int c(b)\mathrm{d}\sigma_n(b) = \int c(b)\mathrm{d}\sigma(b) = s, \tag{C.5}$$

$$\lim_n \int (-c'(b-))\mathrm{d}\sigma_n(b) = \int (-c'(b-))\mathrm{d}\sigma(b). \tag{C.6}$$

Let $b_{min} = \min \text{supp}\sigma$ and $b_{max} = \max \text{supp}\sigma$. We have $c(b_{max}) < s < c(b_{min})$. For each n, define $\alpha_n \in [0,1]$ and $\tilde{\sigma}_n \in \Delta(\text{supp}\sigma)$ as

$$\alpha_n \equiv \begin{cases} \frac{c(b_{min})-s}{c(b_{min})-\int c(b)\mathrm{d}\sigma_n(b)}, & \text{if } \int c(b)\mathrm{d}\sigma_n(b) < s, \\ 1, & \text{if } \int c(b)\mathrm{d}\sigma_n(b) = s, \\ \frac{s-c(b_{max})}{\int c(b)\mathrm{d}\sigma_n(b)-c(b_{max})}, & \text{if } \int c(b)\mathrm{d}\sigma_n(b) > s, \end{cases}$$

and

$$\tilde{\sigma}_n \equiv \begin{cases} \alpha_n \circ \sigma_n + (1-\alpha_n) \circ \delta_{b_{min}}, & \text{if } \int c(b)\mathrm{d}\sigma_n(b) < s, \\ \sigma_n, & \text{if } \int c(b)\mathrm{d}\sigma_n(b) = s, \\ \alpha_n \circ \sigma_n + (1-\alpha_n) \circ \delta_{b_{max}}, & \text{if } \int c(b)\mathrm{d}\sigma_n(b) > s, \end{cases}$$

where δ_b denotes the strategy that puts probability one on cutoff $b \in \{b_{min}, b_{max}\}$. By construction, $\int c(b)\mathrm{d}\tilde{\sigma}_n(b) = s$. Therefore, each $(\tilde{\sigma}_n, v)$ is an equilibrium under c, and the corresponding expected profit of a matched firm is still π. Let $b_{e,n}$ be the corresponding total welfare. By (C.5), it is easy to see that $\lim_n \alpha_n = 1$. Thus, $\lim_n \int(-c'(b-))\mathrm{d}\tilde{\sigma}_n(b) = \int(-c'(b-))\mathrm{d}\sigma(b)$ by (C.6). Hence, we have

$$\lim_n(b_{e,n} - v) = \lim_n \frac{\pi}{\int(-c'(b-))\mathrm{d}\tilde{\sigma}_n(b)} = \frac{\pi}{\int(-c'(b-))\mathrm{d}\sigma(b)} = b_e - v.$$

For each $k > 1$, pick n_k large enough so that $b_{e,n_k} > b_e - \frac{1}{k}$. Because $\tilde{\sigma}_{n_k}$ is finitely supported, by Claim C.4, there exist a_k and $b_k \geq b_{e,n_k}$ such that $c^s_{a_k,b_k,v}$ is feasible. Taking subsequence if necessary, assume $\lim_k a_k = a$ and $\lim_k b_k = b$. Clearly, we have $b \geq b_e$. It is then routine to verify that $c^s_{a,b,v}$ is feasible.[26] This completes the proof. $\square$

[25] Over the whole interval $(0, 1]$, $c'(b-)$ may not be continuous. However, it is continuous over $\text{supp}\sigma$, since $c'(b-) = -\frac{\pi}{b-v}$ for all $b \in \text{supp}\sigma$.

[26] See, for example, the proof of Proposition 2 in Dogan and Hu (2022).