北京大学中国经济研究中心
China Center for Economic Research
讨论稿系列
Working Paper Series

No.E2016007 April 2016

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Option Pricing with the Realized GARCH Model: An Analytical Approximation Approach*

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August 30, 2016

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*We are grateful to Bob Webb (editor) and all participates of the 1st China Derivatives Markets Conference (CDMC) and the 6th IMS-FIPS Workshop for useful comments that substantially improved the paper. Zhuo Huang acknowledges financial support from the Youth Fund of the National Natural Science Foundation of China (71201001,71671004). Tianyi Wang (corresponding author) acknowledges financial support from the Youth Fund of National Natural Science Foundation of China (71301027), the Ministry of Education of China, Humanities and Social Sciences Youth Fund (13YJC790146) and the Fundamental Research Fund for the Central Universities in UIBE(14YQ05). Peter Reinhard Hansen acknowledges support from the Center for Research in Econometric Analysis of Time Series (DNRF78), funded by the Danish National Research Foundation. Corresponding email: tianyiwang@uibe.edu.cn.

1 Introduction

It is well known that realized measures of volatility, which are computed from high frequency data, provide accurate measurements of the latent volatility process. The prime example is the realized variance, see e.g. Andersen et al. (2003). Volatility is fundamental for option pricing, so it is natural to explore ways to incorporate realized measures into option pricing. Several papers have recently shown that discrete time models that incorporate the realized variance, can significantly improve the performance of option pricing. For example, Christoffersen et al. (2014) develop an affine discrete-time model to provide a closed-form option valuation formula through the conditional moment-generating function. The volatility dynamic is modeled as a weighted average between components from daily returns and realized variances, where both components have a Heston-Nandi[1] structure. They show that including the realized variance results in a considerable pricing improvement. Their paper also suggests the need for further research on pricing, using non-affine models and modeling leverage effect separately for both return and realized measures. A related framework is that in Corsi et al. (2013), who employ a Heterogeneous Autoregressive Gamma (HARG) model. This model assumes that the realized variance follows a simple process (with linear long-memory features) and option pricing can be obtained using Monte Carlo simulation. This model was further developed in Majewskia et al. (2015), who enhance the HARG model with a Heston-Nandi type leverage. Their framework includes a class of linear GARCH models with parabolic leverage, including those in Heston and Nandi (2000) and Christoffersen et al. (2008), and the framework conveniently leads to a closed-form option pricing formula.

In this paper, we derive the option pricing formula for the Realized GARCH framework, which may result in better pricing performance, becauce the Realized GARCH framework has proven to be superior to conventional GARCH models for the modeling of returns and for forecasting volatility. The Realized GARCH model was proposed by Hansen et al. (2012), and further refined by Hansen and Huang (2016), which is the variant we adopt for the option pricing in this paper. The model may be labelled as a non-affine log-linear Realized Exponential GARCH model.

The Realized GARCH framework is attractive for option pricing for several reasons. First, the realized variance is incorporated in the model and linked to the latent conditional volatility through a measurement equation. This not only improves the accuracy of the volatility forecast, but also allows for an additional risk premium that relates to volatility-specific shocks. Second, the model benefits from having both return and volatility shocks, similar to stochastic volatility models. Still, the Realized GARCH model is an observation-driven model that permit straight forward estimation by the maximum likelihood. Third, the measurement equation in our model does not require the realized measure to be an unbiased estimator of the daily volatility. Unbiased estimators are difficult to obtain because high-frequency data is only available for a fraction of the day. Market microstructure noise that is not properly accounted for, see

[1] In the Heston-Nandi GARCH model (Heston and Nandi (2000)), volatility is filtered through return data and a closed-form pricing formula is provided.

Hansen and Lunde (2006), can also induce bias in realized measures. In contrast, the HARG model requires an unbiased estimator, and assumes that a rescaling of the realized variance (based on the trading hour's information) achieves this objective. Fourth, the model operates with distinct leverage functions for returns and realized variances. The importance of this flexibility is documented in Hansen et al. (2016) for the pricing of the CBOE volatility index (VIX). Fifth, the log-linear specification avoids many of the constraints that often must be imposed to guarantee positivity of the volatility process. Taking the logarithm also serves to reduce the impact of outliers in the realized measure of volatility. This transformation makes the model more stable, especially during periods with high volatility of volatility.

A quasi closed-form option pricing formula is very hard to obtain for a non-affine model, using the standard method based on the moment-generating function and the inverse Fourier transformation. When a closed-form expression is unavailable one can resort to Monte Carlo methods, see e.g. Corsi et al. (2013) and Kanniainen et al. (2014). While this method is straight forward to apply, it can be very time consuming to achieve a desirable accuracy. This makes analytical approximation methods an attractive alternative. For GARCH-type models, including non-affine models, Duan et al. (1999) and Duan et al. (2006), developed an analytical approximation method that is based on a Gram-Charlier series expansion, which is closely related to the Edgeworth expansion used in this paper. The basic idea is to expand the density of the cumulative return with its first four moments and a standard normal density. Although the closed-form pricing formula is not available, moments of the cumulative return can be calculated with analytical formulas. Compared to the Monte Carlo simulation, an analytical approximation is much faster in practice and free of sampling errors, but can suffer from approximation error. In our Realized GARCH framework, the approximation error is indeed problematic for the Gram-Charlier based method of Duan et al. (1999). We therefore derive an analytical approximation method based on an Edgeworth expansion, which performs much better in simulations and empirically.

Gram-Charlier and Edgeworth expansions are similar, which may explain that Gram-Charlier is sometimes incorrectly labeled as Edgeworth. This mislabeling is used in much of the related literature, including Jarrow and Rudd (1982), Duan et al. (1999) and Duan et al. (2006). Despite their similarity, the two expansions employ different truncations which results in different properties.[2] Cramér compared the two expansion in a series of papers, and deemed the Edgeworth expansion to be superior, see e.g. Cramér (1946). Formally, when a Gaussian distribution is employed as reference distribution, Blinnikov and Moessner (1998) show that the Edgeworth expansion achieves a better approximation for near-Gaussian distribution, and Eggers et al. (2011) highlights how poorly the Gram-Charlier expansion is at approximating a symmetric Normal Inverse Gaussian distribution, while the Edgeworth expansion performs well.

[2]The density of a sum of n random variable may be approximated by an truncated expansion involving a reference density. The order of the Gram-Charlier expansion is defined from the number of included derivatives of the reference density. F. Y. Edgeworth observed that successive terms are not of decreasing order of magnitude in terms of powers of n. Sorting and collecting terms with the same power of n, $n^{1/2}$, $n^{2/2}, \ldots$, and including all terms up to order $n^{k/2}$ defines the Edgeworth expansion (of order k).

There are three common approaches to estimating an option-pricing volatility model.[3] The first approach is to estimate the volatility model for returns, e.g. by maximizing the log-likelihood, see e.g. Christoffersen et al. (2003), then estimate equity risk parameters by no-arbitrage conditions. For a GARCH-type model, which has a single shock, the equity risk premium parameter may be identified by matching expected return to the risk-free rate (under the risk-neutral measure). This approach to estimation is not directly applicable to the Realized GARCH framework, because it has two separate shocks and it would require an additional moment condition to identify both the equity risk premium parameter and the volatility risk premium parameter. A second approach to estimation in this context, is to directly target option pricing, and estimate all parameters by minimizing the option pricing error[4], see e.g. Heston and Nandi (2000). This method obviously delivers the best in-sample fit in terms of option pricing. However, it tends to result in absurd parameter estimates and large out-of-sample pricing error. A third estimation approach is to maximizing a joint "likelihood", where parameters are estimated to explain underlying time series, (e.g. returns and realized measures) as well as observed option prices.[5] (see Christoffersen et al. (2014) etc.). This method, that entails a trade-of between fitting the underlying time series and option prices, has received increasing attention in the option pricing literature.

In this paper, we conduct an extensive empirical analysis with a large panels of option prices. The data set spans fourteen years of which twelve are used for in-sample estimation and two years are used for out-of-sample evaluation. We consider a range of distinct models in our comparisons, including the Heston-Nandi GARCH by Heston and Nandi (2000) (which is commonly used benchmark model); two linear asymmetric models: NGARCH (Engle and Ng (1993)) and GJR-GARCH (Glosten et al. (1993)); a log-linear asymmetric model: EGARCH (Nelson (1991)) and the recently proposed GARCH model with realized variance: GARV (Christoffersen et al. (2014)). The main conclusion is that the use of realized measures greatly reduces option pricing errors, and our Edgeworth-based pricing formula in conjunction with the Realized GARCH model has be best out-of-sample performance. For instance, the Realized GARCH model reduces the out-of-sample pricing errors by 18.9% on average, relative to those of the GARV model, and by 22.3% or more relative to all other models in our comparison. For option based parameters, the two figures are nearly 30%[6]. Those results highlight the empirical gains on a non-affine model with separate leverage effect and reinforce the existing literature on the importance of including realized variance into option pricing.

The remainder of this paper is organized as follows. In Section 2, we provide a brief introduction of the Realized GARCH model and the corresponding risk neutralization

[3]See Christoffersen et al. (2013) for a detailed explanation.

[4]Usually in terms of the difference in implied volatility or other equivalent measures such as the Vega weighted pricing error.

[5]Or fitting error for other Q-measure series such as the volatility index (see Kanniainen et al. (2014) etc.).

[6]The provided out-of-sample results are based on the extended 2 years of option data. We also provide an alternative out-of-sample comparison with Thursday data (parameters are estimated with Wednesday data, see main text for details), and the four figures are 4.3%, 14.2%, 10.9%, and 18.1%.

procedure. In Section 3, we discuss how to price European Call options with an analytical approximation. Special attention is paid to the expansion where the proper terms for the Edgeworth expansion are derived. In Section 4, we present all the competing models used in our comparisons. Empirical results are presented in Section 5, where we compare the models, both in-sample and out-of- sample, and with a variety of estimation methods. Pricing performance is evaluated at the aggregated level and for different subcategories in terms of the moneyness, maturity and volatility index level. The final section concludes and provides two directions for further research.

2 The Model

2.1 Realized GARCH model

The Realized GARCH model we adopt in this paper is given by:

$$\begin{aligned} r_{t+1} &= r + \lambda\sqrt{h_{t+1}} - \tfrac{1}{2}h_{t+1} + \sqrt{h_{t+1}}z_{t+1}, & (1)\\ \log h_{t+1} &= \omega + \beta \log h_t + \tau_1 z_t + \tau_2(z_t^2 - 1) + \gamma\sigma_u u_t, & (2)\\ \log x_t &= \xi + \phi \log h_t + d_1 z_t + d_2(z_t^2 - 1) + \sigma_u u_t, & (3) \end{aligned}$$

where z_t and u_t are independent standard normal random variables. This is the variant proposed by Hansen and Huang (2016) that has an explicit leverage term in its GARCH equation (2). See Hansen et al. (2012) for additional variants of the Realized GARCH model. Hansen et al. (2016) model the CBOE VIX with this model and find that an additional leverage term in addition to the realized variance can deliver a better fit and forecast of the physical dynamic of S&P500, the risk neutral dynamic of VIX, and the volatility risk premium (measured by the different between the two).

This model ahs two characteristics that are attractive in the present context for option pricing. First, the Realized GARCH model shares a key feature of stochastic volatility model, in having an innovation term, u_t, that relate directly to volatility, but the model is much easier to be estimated, because it is an observation driven model. Second, the Realized GARCH model, as formulated above, has fewer parameter restrictions than related discrete time models that also utilize realized variance in option pricing. This also simplifies the estimation.

2.2 Risk neutralization

To derive the option pricing formula, we need to find the model dynamics under the risk neutral measure. The following exponentially affine stochastic discount factor (SDF) is used for risk neutralization,

$$Z_{t+1} = \frac{\exp(v_{1,t}z_{t+1} + v_{2,t}u_{t+1})}{\mathbb{E}(\exp(v_{1,t}z_{t+1} + v_{2,t}u_{t+1}))} = \exp\left(v_{1,t}z_{t+1} + v_{2,t}u_{t+1} - \frac{v_{1,t}^2}{2} - \frac{v_{2,t}^2}{2}\right),$$

where z_t and u_t are independent standard normal random variables.

This specification of risk-neutralization has also been adopted in option pricing models using realized variance, such as those of Corsi et al. (2013), Christoffersen et al. (2014)

and Majewski et al. (2015). Such specification of risk-neutralization degenerated to the locally risk-neutral valuation relationship (LRNVR) of Duan (1995) when $u_t = 0$.

Let $R_{t+1} = y_{t+1} - y_t$. The non-arbitrage condition, $\mathbb{E}_t^Q(\exp(R_{t+1})) = \exp(r)$, yields

$$\begin{aligned}\mathbb{E}_t^Q(\exp(R_{t+1})) &= \mathbb{E}_t(Z_{t+1}\exp(R_{t+1})) \\ &= \exp(r + \lambda\sqrt{h_{t+1}} + v_{1,t}\sqrt{h_{t+1}}) = \exp(r),\end{aligned}$$

with the implication that

$$v_{1,t} = -\lambda.$$

Therefore, the risk neutral moment-generating function is

$$\begin{aligned}\mathbb{E}_t^Q(\exp(s_1 z_{t+1} + s_2 u_{t+1})) &= \mathbb{E}_t(Z_{t+1}\exp(s_1 z_{t+1} + s_2 u_{t+1})) \\ &= \exp\left(-s_1\lambda + s_2 v_{2,t} + \frac{s_1^2}{2} + \frac{s_2^2}{2}\right).\end{aligned}$$

This implies, that we under the risk neutral Q-measure have

$$\begin{aligned}z_{t+1}^* &= z_{t+1} + \lambda, \\ u_{t+1}^* &= u_{t+1} - v_{2,t}.\end{aligned}$$

Here, we set $v_{2,t} = \chi$ and assume it is time invariant to ensure that the model is also affine under the risk neutral measure.

Hence, the dynamics under the Q-measure is

$$r_{t+1} = r - \tfrac{1}{2}h_{t+1} + \sqrt{h_{t+1}}z_{t+1}^*, \tag{4}$$

$$\log h_{t+1} = \omega + \beta\log h_t + \tau_1(z_t^* - \lambda) + \tau_2((z_t^* - \lambda)^2 - 1) + \gamma\sigma_u(u_t^* + \chi), \tag{5}$$

$$\log x_t = \xi + \phi\log h_t + d_1(z_t^* - \lambda) + d_2(((z_t^* - \lambda)^2 - 1) + \sigma_u(u_t^* + \chi), \tag{6}$$

where z_t^* and u_t^* are independent standard normal random variables.

When we estimate the model under the Q-measure, the effective model is

$$\begin{aligned}r_{t+1} &= r - \tfrac{1}{2}h_{t+1} + \sqrt{h_{t+1}}z_{t+1}^* \\ \log h_{t+1} &= \omega^* + \beta\log h_t + \tau_1(z_t^* - \lambda) + \tau_2((z_t^* - \lambda)^2 - 1) + \gamma\sigma_u u_t^*, \\ \log x_t &= \xi^* + \phi\log h_t + d_1(z_t^* - \lambda) + d_2((z_t^* - \lambda)^2 - 1) + \sigma_u u_t^*.\end{aligned}$$

Note that if we estimate the model with option data based nonlinear least squares, the variance σ_u cannot be identified and the effective number of parameters is reduced by 2. When our model is estimated with the joint estimation, the likelihood of the underlying time-series helps to identify σ_u and χ separately.

3 Analytical Approximation of a European call

3.1 Edgeworth expansions

The Realized GARCH model does not have an analytical formula for the moment generating function. That means the conventional explicit pricing formula via Fourier inverse transformation is not applicable. Instead, we use an analytical approximation method to calculate the option price. The more straightforward Monte Carlo simulation method is not used because we want to calibrate key parameters with the large scale option data set. The Monte Carlo method does not provide a good foundation for this task as it is less time efficient and subject to substantial sampling errors.

To get the formula, we expand the density of the cumulative return using the following second-order Edgeworth expansion.[7]

$$g(z) = \left[1 + \frac{\kappa_3}{6}H_3(z) + \frac{(\kappa_4 - 3)}{24}H_4(z) + \frac{\kappa_3^2}{72}H_6(z)\right]\phi(z),$$

where $\phi(z)$ is the density function of the standard normal distribution, and $H_n(z)$ is the n-th order (probabilist's) Hermite polynomial, given by

$$\begin{aligned} H_3(z) &= z^3 - 3z, \\ H_4(z) &= z^4 - 6z^2 + 3, \\ H_6(z) &= z^6 - 15z^4 + 45z^2 - 15. \end{aligned}$$

The Gram-Charlier expansion, used in Duan et al. (1999), employs a different truncation, which causes it to exclude the term, $\frac{\kappa_3^2}{72}H_6(z)$, from the expression above. We find the Gram-Charlier expansion to be inadequate in the present context. To illustrate this point, we use parameters estimated for the Realized GARCH model from Table 3 to calculate the analytical approximation of moments for 90 days horizon. Those moments are then used to compute the analytical approximations to cumulative returns. Additionally, we conduct a Monte Carlo simulation to calculate the corresponding empirical density. Those two densities are plotted in Figure 1. We provide the corresponding results for the conventional EGARCH in Figure 2.

[Insert Figure 1 and Figure 2 here]

From Figure 1 it is evident that the approximation based on the Gram-Charlier expansion is inadequate for the Realized GARCH model. A significant improvement is obtained by using the Edgeworth expansion instead. From Figure 2 we observe the that Gram-Charlier performs reasonably well in this design where the underlying model is the EGARCH model, although the Edgeworth expansion continue to be a bit more accurate. A key difference between these the Realized GARCH model and the EGARCH model,

[7]The Edgeworth expansion is designed to approximate the distribution of the standardized sum of i.i.d. random variables $S_n = \sum_{i=1}^{n}(X_i - \mu_x)/\sqrt{n\sigma_x^2}$. A Taylor expansion of the characteristic function of S_n yields the following first three terms of the expansion are: $\phi(z)$, $\frac{\kappa_3}{6}H_3(z)\phi(z)$, $\left(\frac{\kappa_4-3}{24}H_4(z) + \frac{\kappa_3^2}{72}H_6(z)\right)\phi(z)$, where κ_i is the i-th moment and $\phi(z)$ is the pdf of the standardized normal distribution.

is that the former has an additional innovation shock – the volatility shock u. This shock has important implication for the distribution. For instance it generates fatter tails which the Edgeworth expansion does a better job at capturing. For this reason we will deduce the option price approximation from an Edgeworth expansion, rather than the Gram-Charlier which is commonly used in this literature.

3.2 Pricing formula

Under the risk neutral measure, the European call option price is given by:

$$e^{-rT}\mathbb{E}_0^Q(\max(S_T-K,0)).$$

Let $R_T=\log(S_T/S_0)$, as shown in the appendix, the expectation can be written as an integral of the standardized cumulated return $z_T=(R_T-\mu)/\sigma$:

$$e^{-rT}\int_{-\infty}^{k}\left[S_0\exp\left(\mu-\sigma z\right)-K\right]\tilde{g}(z)\mathrm{d}z,$$

where $k=(\log(S_0/K)+\mu)/\sigma$ and $\tilde{g}(z)=g(-z)$. By inserting the expansion formula and integrate we obtain the pricing formula given in the following Proposition.

Proposition 1. *The price of a European call option associated with the Realized GARCH model can be expressed as*

$$C_{approx}=C+\kappa_3A_3+(\kappa_4-3)A_4+\kappa_3^2A_6, \tag{7}$$

where:

$$\begin{aligned}
C &= S_0e^{\delta\sigma}\Phi\left(d\right)-Ke^{-rT}\Phi\left(d-\sigma\right),\\
A_3 &= \frac{1}{6}S_0e^{\delta\sigma}\sigma\left[\left(2\sigma-d\right)\phi\left(d\right)+\sigma^2\Phi\left(d\right)\right],\\
A_4 &= \frac{1}{24}S_0e^{\delta\sigma}\sigma\left[\left(d^2-1-3\sigma\left(d-\sigma\right)\right)\phi\left(d\right)+\sigma^3\Phi\left(d\right)\right],\\
A_6 &= \frac{1}{72}S_0e^{\delta\sigma}\sigma\left[\sigma^5\Phi(d)+\left(3-6d^2+d^4+5\sigma\left(d-(d-\sigma)(\sigma d-2)-(d-\sigma)^3\right)\right)\phi(d)\right],\\
d &= \frac{\log(S_0/K)+\mu}{\sigma}+\sigma
\end{aligned}$$

where $\kappa_i=\mathbb{E}_0^Q(z_T^i)$*.*

Proof. See appendix. □

To calculate option price from (7), we need to calculate the first four moments of the standardized cumulative return. Since equation (1) provides the non-standardized return, it is more convenient to express κ_i and in terms of $\mathbb{E}_0^Q(R_T^i)$:

$$\kappa_3=\frac{1}{\sigma^3}\left[\mathbb{E}_0^Q\left(R_T^3\right)-\mu^3\right]-3\frac{\mu}{\sigma}\qquad\kappa_4=\frac{1}{\sigma^4}\left[\mathbb{E}_0^Q\left(R_T^4\right)-\mu^4\right]-2\frac{\mu}{\sigma}\left(2\kappa_3+3\frac{\mu}{\sigma}\right).$$

The evaluation of $\mathbb{E}_0^Q(R_T^i)$ is given in the appendix.

4 Competing Models

Five competing models are considered: EGARCH, NGARCH, GJR-GARCH, HN-GARCH and GARV. Except for the EGARCH model, all models are linear. Apart from the HN-GARCH and the GARV model, no models have explicit pricing formula. Only the GARV model utilizes realized variance as additional underlying information. Table 1 provides an overview of competing models. Due to the inclusion of realized variance, the number of parameters is greatly increased compared with traditional GARCH-type models.

[Insert Table 1 here]

More details of each competing model are listed below.

EGARCH

The physical dynamic of the EGARCH model is

$$\begin{aligned} r_{t+1} &= r + \lambda\sqrt{h_{t+1}} - \tfrac{1}{2}h_{t+1} + \sqrt{h_{t+1}}z_{t+1}, \\ \log h_{t+1} &= \beta_0 + \beta_1 \log h_t + \tau_1 z_t + \tau_2\left(|z_t| - \sqrt{\tfrac{2}{\pi}}\right). \end{aligned}$$

The risk neutral counterpart is

$$\begin{aligned} r_{t+1} &= r - \tfrac{1}{2}h_{t+1} + \sqrt{h_{t+1}}z^*_{t+1} \\ \log h_{t+1} &= \beta_0 + \beta_1 \log h_t + \tau_1\left(z_t - \lambda\right) + \tau_2\left(|z_t - \lambda| - \sqrt{\tfrac{2}{\pi}}\right) \end{aligned}$$

The persistence parameter under both measures is identical.i.e. $\pi^P = \pi^Q = \beta_1$.

NGARCH

The physical dynamic of the NGARCH model is

$$\begin{aligned} r_{t+1} &= r + \lambda\sqrt{h_{t+1}} - \tfrac{1}{2}h_{t+1} + \sqrt{h_{t+1}}z_{t+1}, \\ h_{t+1} &= \beta_0 + \beta_1 h_t + \tau_1 h_t\left(z_t - \tau_2\right)^2. \end{aligned}$$

The risk neutral counterpart is

$$\begin{aligned} r_{t+1} &= r - \tfrac{1}{2}h_{t+1} + \sqrt{h_{t+1}}z^*_{t+1}, \\ h_{t+1} &= \beta_0 + \beta_1 h_t + \tau_1 h_t\left(z^*_t - (\tau_2 + \lambda)\right)^2. \end{aligned}$$

The persistence parameter is $\pi^P = \beta_1 + \tau_1\left(1 + \tau_2^2\right)$ for the physical measure and $\pi^Q = \beta_1 + \tau_1\left(1 + (\tau_2 + \lambda)^2\right)$ for the risk neutral measure.

GJR-GARCH

The physical dynamic of the GJR-GARCH model is

$$\begin{aligned} r_{t+1} &= r + \lambda\sqrt{h_{t+1}} - \tfrac{1}{2}h_{t+1} + \sqrt{h_{t+1}}z_{t+1}, \\ h_{t+1} &= \beta_0 + h_t\left[\beta_1 + \tau_1 z_t^2 + \tau_2 \max(0, -z_t)^2\right]. \end{aligned}$$

The risk neutral counterpart is

$$\begin{aligned} r_{t+1} &= r - \tfrac{1}{2}h_{t+1} + \sqrt{h_{t+1}}z^*_{t+1}, \\ h_{t+1} &= \beta_0 + h_t\left[\beta_1 + \tau_1(z_t-\lambda)^2 + \tau_2 \max(0, -(z_t-\lambda))^2\right]. \end{aligned}$$

The persistence parameter $\pi^P = \beta_1 + \tau_1 + \frac{\tau_2}{2}$ for the physical measure and $\pi^Q = \beta_1 + (\tau_1 + \tau_2\Phi(\lambda))\left(1+\lambda^2\right) + \tau_2\lambda\phi(\lambda)$ for the risk neutral measure.

HN-GARCH

The physical dynamic of the HN-GARCH model is

$$\begin{aligned} r_{t+1} &= r + \left(\lambda - \tfrac{1}{2}\right)h_{t+1} + \sqrt{h_{t+1}}z_{t+1}, \\ h_{t+1} &= \beta_0 + \beta_1 h_t + \tau_1\left(z_t - \tau_2\sqrt{h_t}\right)^2. \end{aligned}$$

The risk neutral counterpart is

$$\begin{aligned} r_{t+1} &= r - \tfrac{1}{2}h_{t+1} + \sqrt{h_{t+1}}z^*_{t+1}, \\ h_{t+1} &= \beta_0 + \beta_1 h_t + \tau_1\left(z^*_t - (\tau_2+\lambda)^2\sqrt{h_t}\right). \end{aligned}$$

The persistence parameter is $\pi^P = \beta_1 + \tau_1\tau_2^2$ for the physical measure and $\pi^Q = \beta_1 + \tau_1\left(\lambda+\tau_2\right)^2$ for the risk neutral measure.

GARV

The physical dynamic of the GARV model is

$$\begin{aligned} r_{t+1} &= r + \left(\lambda - \tfrac{1}{2}\right)\bar{h}_{t+1} + \sqrt{\bar{h}_{t+1}}z_{t+1}, \\ \bar{h}_{t+1} &= \kappa h^R_{t+1} + (1-\kappa)h^{RV}_{t+1}, \\ h^R_{t+1} &= \omega + \beta h^R_t + \tau_1\left(z_t - \tau_2\sqrt{\bar{h}_t}\right)^2, \\ RV_t &= h^{RV}_t + \alpha_2\left[(\epsilon_t - d_2\sqrt{\bar{h}_t})^2 - (1 + d_2^2\bar{h}_t)\right], \\ h^{RV}_{t+1} &= \xi + \phi h^{RV}_t + d_1\left(\epsilon_t - d_2\sqrt{\bar{h}_t}\right)^2, \end{aligned}$$

where (z_t, u_t) follows a standard bivariate normal distribution with $\rho_{z\epsilon} = \rho$. We also introduce $\gamma = d_1/\alpha_2$.

The risk neutral dynamics are

$$
\begin{aligned}
r_{t+1} &= r - \tfrac{1}{2}\bar{h}_{t+1} + \sqrt{\bar{h}_{t+1}} z_{t+1}, \\
\bar{h}_{t+1} &= \kappa h^R_{t+1} + (1-\kappa) h^{RV}_{t+1}, \\
h^R_{t+1} &= \omega_1 + \beta_1 h^R_t + \tau_1 \left(z_t - \tau_2^{*2} \sqrt{\bar{h}_t} \right)^2, \\
RV_t &= h^{RV}_t + \alpha_2 \left[(\epsilon_t - d_2^{*2} \sqrt{\bar{h}_t})^2 - (1 + d_2^{*2} \bar{h}_t) \right], \\
h^{RV}_{t+1} &= \xi + \phi h^{RV}_t + d_1 \left(\epsilon_t - d_2^{*2} \sqrt{\bar{h}_t} \right)^2.
\end{aligned}
$$

The persistence parameter for the GARV model is a vector. To make things simpler, we define the persistence parameter as the maximum of the two persistence parameters associated with h^R and h^{RV}. i.e. $\pi^P = \beta_1 + \tau_1 \tau_2^2 \kappa$ for the physical measure and $\pi^Q = \beta_1 + \tau_1 \tau_2^{*2} \kappa$ for the risk neutral measure.

5 Empirical Results

5.1 Data

The empirical comparisons are based on daily returns of the S&P500 index, the realized variance as well as the daily option data[8]. As the realized variance starts at 200001 and the option data ends at 201412, we use the option data from 2000-2012 for in-sample comparison and reserve 2013-2014 for out-of-sample evaluation. The option data are trimmed using the following common method:

1. Keep option data on Wednesday (for in-sample) and Thursday (for out-of-sample) only.
2. Drop all options with zero (daily) trading volume and missing implied volatility.
3. Calculate option price with the average of best bid and best ask and drop all options with price less than 5 dollar.
4. Drop all options whose time to maturity is less than 15 days or longer than 180 days.
5. Drop in-the-money options and very deep out-of-the-money options (i.e. $S/K > 1.3$ or $S/K < 0.7$).[9]
6. Keep the six most liquid (in terms of daily trading volume) for every maturity in each day.
7. Replace put options with call options using the put-call parity.

Table 2 provides an extensive summary for the number of contracts and the average implied volatility within each subcategory in our option data set. Panel A summaries data which will be used to estimate parameters. Panel B and C summarize data that will

[8]These data are collected from Yahoo finance, the Realized Library at the Oxford-Man institute, and OptionMetrics from WRDS.

[9]In Corsi et al. (2013), deep out-of-momey is defined as $S/K > 1.06$ or $S/K < 0.94$.

be used to implement out-of-sample comparison. Following Christoffersen et al. (2014), we use Black-Scholes delta to measure the moneyness of options. As we only keep out-of-the-money options, those with deltas higher than 0.7 are deep out-of-the-money put options. It can be seen that those options are relatively expensive compared with out-of-the-money calls which display the stylized volatility smirk across moneyness. The smirk pattern is more profound in recent years.

[Insert Table 2 here]

5.2 Estimation method

We estimate a model with joint likelihood of the observed time series and the pricing errors, where the latter are weighted by the vega[10]. Unlike traditional calibration method focusing only on pricing errors, this approach also takes the model's ability to replicate underlying dynamics into account. The parameter to be estimated for option pricing with the Realized GARCH model are given by:

$$\Theta = \{\lambda, \omega, \beta, \tau_1, \tau_2, \gamma, \xi, \phi, d_1, d_2, \sigma_u, \log h_1, \chi, \sigma_e\}$$

where σ_e is the standard deviation of vega-weighted option pricing errors. With the data $\{ret_t, rv_t | t = 1, 2, ..., T\}$ and $\{option_i | i = 1, 2, ..., N\}$ the parameters can be estimated by maximizing the joint likelihood function:

$$\ell_{full}(ret, rk, option; \Theta) = \ell_r(ret; \Theta) + \ell_x(rv; \Theta, ret) + \ell_o(option; \Theta, ret, rv)$$

where:

$$\begin{aligned}
\ell_r &= -\frac{T}{2}\log(2\pi) - \tfrac{1}{2}\sum_{t=1}^{T}\log(h_t) - \sum_{t=1}^{T}\frac{(ret_t - r - \lambda\sqrt{h_t} + \frac{1}{2}h_t)^2}{2h_t}, \\
\ell_x &= -\frac{T}{2}\log(2\pi) - \frac{T}{2}\sum_{t=1}^{T}\log(\sigma_u^2) - \sum_{t=1}^{T}\frac{(\log rv_t - \xi - \phi\log h_t - d_1 z_t - d_2(z_t^2 - 1))^2}{2\sigma_u^2}, \\
\ell_o &= -\frac{N}{2}\log(2\pi) - \frac{N}{2}\sum_{i=1}^{N}\log(\sigma_e^2) - \sum_{i=1}^{N}\frac{\left(\left(P_i^{Mod} - P_i^{Mkt}\right)/\nu_i\right)^2}{2\sigma_e^2},
\end{aligned}$$

where P_i^{Mod} and P_i^{Mkt} are the model-implied option price and the market price, respectively. The weighting-parameter, ν_i, is the Black-Scholes Vega that measures the option's sensitivity to implied volatility changes. $\left(P_i^{Mod} - P_i^{Mkt}\right)/\nu_i$ is, therefore, an approximation of the difference in implied volatility. From the first order conditions of the likelihood, it follows that σ_e^2 is simply estimated by $\frac{1}{N}\sum_{i=1}^{N}\left(\left(P_i^{Mod} - P_i^{Mkt}\right)/\nu_i\right)^2$, and we note that its square root, σ_e is identical to a term labelled νRMSE in Kanniainen et al. (2014).

An alternative way to formulate joint estimation is focusing on the VIX index instead of option price. We do not use this method because it ignores key information for maturities other than 30 days (in our case, the maturity varies from 15 to 180).

[10] The analytical approximation (7) ensures that the Vega-adjusted option pricing error is free from sampling error and therefore a well defined objective function.

Information on variance risk premium over different time horizons might not be fully conveyed by the volatility index. Using a term structure for VIX might be a solution to this problem. However, constructing the historical VIX term structure itself calls for considerable effort.

We also calibrate models using the conventional nonlinear least square (NLS) method, where only the ℓ_o is considered. Although both return and realized variance are used in the calculation, their only job is filtering the latent volatility. The goodness of fit on both series is irrelevant. Approximation formulas for EGARCH, NGARCH and GJR-GARCH models are provided in Duan et al. (1999) and Duan et al. (2006). We follow their results when options are priced under corresponding models[11].

5.3 Parameter estimations

Table 3 provides parameters estimated from both methods for all models considered. Robust standard errors are in parenthesis. Likelihood function values and persistence parameters of volatility dynamics are also provided.[12]

[Insert Table 3 here]

The left columns are associated with joint estimation where fit of option price and underlying dynamics are both taken into account. For the Realized GARCH model, we have: 1) β close to one indicating a strong persistence of volatility dynamics; 2) a significant leverage effect indicated by τ_1, τ_2, d_1 and d_2; 3) ϕ close to 1, reinforcing the idea that (3) is a measurement equation; 4) a significant contribution of realized volatility through γ, indicating the importance of realized measures; 5) significant equity premium (λ) and volatility premium (χ); 6) the smallest σ_e that is a measure of (ν-weighted) option pricing error.

We estimate the GARV model in two steps because the stationary constraint presented in Christoffersen et al. (2014) cannot guarantee a positive ω_1 and ξ. First, we estimate the model with the constraints used for NLS estimation to make sure that the volatility is positive. Second, we re-estimate τ_1^* and τ_2^* with stationary constraints.[13] As parameters τ_1, τ_2, τ_1^* and τ_2^* driven to their limits (almost to the boundary defined by the constraints), the volatility risk premium is very small and the σ_e is a little larger than the Realized GARCH model. Models without realized measures all have significantly larger option pricing error, as defined by σ_e. We also find that π^Q is generally larger than π^P, which confirms findings in the option pricing literature that volatility is more persistent under the risk neutral measure.

[11] Kanniainen et al. (2014) use NGARCH and GJR-GARCH with NLS estimation where price is calculated using a Monte Carlo simulation. This method is slow and can generate different prices for the same option even with the same set of parameters (due to sampling error) without holding the seed of the random variable generator constant.Therefore, we use approximation method instead.

[12] For joint estimation, the persistence parameters under the physical and risk neutral measures are provided. For option-based estimation, only the risk neutral persistence parameters are provided.

[13] We would like to thank Peter Christoffersen for his kindness in sharing with us the NLS estimation code for the GARV model. We also estimated the model with NLS constraints on τ_1, τ_2 and stationary constraints on τ_1^*, τ_2^* together. Although this method can improve σ_e, it delivers strangely small τ_1,τ_2 and significantly larger out-of-sample σ_e (probability due to in-sample over-fitting).

The right columns are associated with NLS estimation where the objective solely focuses on fitting option price. This empirical results with this estimation method leads the the following observations. First, the persistence parameter is larger than that obtained with joint estimation. Second, the parameters in the Realized GARCH model are substantially different, such as those related to the leverage effect (significantly more asymmetry) and γ (now smaller), which measures that impact that the realized measure has on volatility. We also observe that the estimated ϕ in the measurement equation now differs substantially from unity. For the GARV model we now estimate the correlation between two shocks to be one. Some of this estimates are obviously unrealistic, but are chosen by the NLS procedure because these values help minimize the in-sample option pricing errors. Third, the Realized GARCH model outperforms the other models. Perhaps impressively, the Realized GARCH model with joint estimation does better than the GARV model with NLS estimation in terms of IVRMSE.

It is worth noting that the QMLE estimation method used for the GARV model assumes that the return and realized variance follows a bivariate normal distribution, which is clearly at odds with reality. The values of the likelihood functions of the Realized GARCH model and the GARV model are therefore not directly comparable.

5.4 In-sample pricing performance

Table 4 provides detailed in-sample pricing performance for both methods. From here on, we evaluate the model's pricing performance through the root mean square of implied volatility (IVRMSE):

$$IVRMSE = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left[IV_i^{Mod} - IV_i^{Mkt}\right]^2} \times 100,$$

where IV_i^{Mod} and IV_i^{Mkt} are the Black-Scholes implied volatility of option i calculated with the model price and the market price respectively. The total IVRMSE shows the same patten as σ_e in Table 3. Here, we decompose the total IVRMSE into subcategories according to three different characteristics. The first is the moneyness, which is linked to the model's ability to generate enough leverage effect. The second is the time to maturity, which is linked to the model's ability to track long-run dynamics. The last characteristic is the contemporary volatility index level, which is linked to the model's ability to generate enough variance risk premium. The left columns are results from joint estimation and the right columns are results from the NLS estimation.

[Insert Table 4 here]

Joint estimation

The total IVRMSE shows that the Realized GARCH model has the smallest pricing error and the GARV model has a slightly higher (2.8%) pricing error. The NGARCH and GJR-GARCH models have comparable pricing error, followed by the Heston-Nandi

GARCH model. The EGARCH model has the worst pricing performance. The fact that all models without realized variance have significantly higher IVRMSE highlights the importance of realized measure in option pricing. In subcategory comparison, we still confirm that the Realized GARCH model does a better job in most cases.

It is interesting to compare the Realized GARCH model to the GARV model, which also utilizes realized variance. The Realized GARCH model performs better on deep out-of-the-money options. The GARV model has better performance when the option is less out-of-the-money. This result might be caused by the fact that key parameters of leverage in the GARV model (τ_i, τ_i^*) are limited by the positivity constraints. The Realized GARCH model has better performance at shorter maturity while the GARV model does better at longer maturity. One possible explanation is that the GARV model has the structure of a component volatility model, that are known to excel at modeling long-term volatility. The Realized GARCH model has better pricing ability when the volatility index is higher and the GARV model works better when the volatility index is lower. This can be explained by the fact that the log-linear specification of the Realized GARCH model has an advantage to react to sharp volatility changes through the exponential of $\log h$. It can be seen that the EGARCH model performs worse than all other models. This indicates that log-linear specification needs accurate information on $\log h$, or the measurement error will also be exponentialized and severely jeopardize the model's pricing performance.

NLS estimation

The NLS estimation yields similar results. The Realized GARCH model still outperforms other models, with an 9.5% average improvement over the GARV model and 21% (or more) improvement over methods that do not utilize realized measures. Unlike in the joint estimation, the Realized GARCH model now outperforms the GARV model at all moneyness subcategories and the relative performances across maturities as well as volatility levels are improved.

5.5 Out-of-sample pricing performance

Because the Realized GARCH model has far more parameters than the conventional GARCH models, one might worry that its superior in-sample performace is driven by over-fitting of the data. To verify that this is not the case, we proceed with an out-of-sample comparison.

There are several ways of performing an out-of-sample evaluation in the option pricing literature. 1) Estimate parameters with the first several years in the whole sample, keep the parameters fixed and value option prices in the following years (Christoffersen and Jacobs (2004)). 2) Estimate parameters with a rolling window and value options for the next day (Christoffersen and Diebold (2006)). 3) Use Wednesday options to estimate parameters, keep the parameters fixed and value Thursday options within the same time span (Christoffersen et al. (2010)).

In consideration of our sample size and the time consumption of parameter estimation, we use the first and the last methods in our paper:

1. THU2000-2012: Estimate parameters using Wednesday data from 2000/1 to 2012/12 and calculate price for Thursday options from 2000/1 to 2012/12.
2. WED2013-2014: Estimate parameters using Wednesday data from 2000/1 to to 2012/12 and calculate price for Wednesday options from 2013/1 to 2014/12.

The first method is not exactly a full out-of-sample method as it involves underlying information from future trading days. It is considered out-of-sample because options in Thursday are not used when minimizing the pricing error. The second method is a full out-of-sample method, as neither options nor underlying information are used during the out-of-sample period.

[Insert Table 5 and 6 here]

Joint estimation

Table 5 provides results based on joint estimation parameters. The left columns are associated with Thursday pricing performance and the right columns are for Wednesday pricing performance.

Because model parameters are estimated with information from the future when we value Thursday options, this comparison is similar to an in-sample fit. The only difference is that the Realized GARCH model has even better relative performance for some subcategories than was the case before. Now the Realized GARCH model also outperforms the GARV model at medium volatility levels, some longer maturities, and less deep out-of-the money options.

For Wednesday pricing performance (the pure out-of-sample period), we find that models with realized measures still have better performance. Improvement of the Realized GARCH model is very significant over other models: an 18.9% IVRMSE reduction compared to the GARV model and at least 22.3% IVRMSE reduction compared with other models. For subcategory comparison, the relative performance of the Realized GARCH model and the GARV model are quite different compared with in-sample and Thursday out-of-sample cases. We find that the Realized GARCH model has better performance than the GARV model over most moneyness and longer maturities. The GARV model has better performance than the Realized GARCH model when the volatility index level is high.

NLS estimation

Again, the NLS results shown in Table 6 confirm what we get from the joint estimation. For Thursday pricing performance, the Realized GARCH model is better than the GARV model with 10.9% total IVRMSE reduction and dominates other models by at least 18.1% total IVRMSE reduction. The GARV model still outperforms the Realized GARCH model in the longest maturities and one low volatility level while the Realized GARCH model works better for the rest of the cases.

For Wednesday pricing performance, the Realized GARCH model is still the best with 30.7% reduction over the GARV model and at least 29.4% reduction over other models in terms of total IVRMSE. We also find slightly larger IVRMSE for the Realized GARCH, GARV and NGARCH models which indicates a possible over-fitting for the NLS method. For subcategories, the Realized GARCH model outperforms the other models in most cases. Compared to the GARV model, the Realized GARCH model has better performance over all moneyness groups, maturities over 30 days, and low volatility levels. This result is slightly better than the case in joint estimation where the GARV model beats the Realized GARCH model in five instead of three subcategories.

6 Conclusions

In this paper, we provide an Edgeworth expansion based analytical approximation option pricing formula for the Realized GARCH model. Unlike existing option pricing models that utilizes realized measures, our model has a non-affine exponential GARCH model structure.

We have pointed out that existing approximations for GARCH models, that are labeled "Edgeworth" are in fact Gram-Charlier approximations. Fortunately, the Edgeworth expansion we have derived for the Realized GARCH model is directly to GARCH models, by using the proper moments, as derived in Duan et al. (1999).

We have used and compared two estimation methods. A nonlinear least squares method whose objective entirely focuses on option pricing, and joint likelihood estimation method that simultaneously fits the dynamic properties of the underlying time series in conjunction with option prices. We compare models and estimation methods in terms of their empirical option pricing performance – in-sample as well as the out-of-sample. While the nonlinear least square method, in some cases, has competitive option pricing errors in terms of IVRMSE, some of the parameter estimates it produces are unrealistic, which leads us to prefer the joint likelihood-based estimation method. One clear conclusion that emerges from both in-sample results and out-of-sample results, is that the inclusion of realized measures in this context is highly advantageous for the option pricing. The Realized GARCH model has the best performance on average.

Our analytical and empirical results suggests some directions for further research, which might first enhance the option pricing performance of the Realized GARCH model. One could introduce Heston-Nandi terms into the Realized GARCH model, with the objective of obtaining a a close-form option pricing formula, thereby avoiding the need for approximations. Another possible direction for future research is to pursue a component-type structure of the Realized GARCH model, which might improve its option pricing performance at longer horizons. Finally, it would be interesting to study whether the use of our Edgeworth approximation, brings an important improvement over the existing Gram-Charlier approximation for conventional GARCH models. This was our experience for the Realized GARCH model, but Figure 2 suggest that the gains may be more modest for conventional GARCH models.

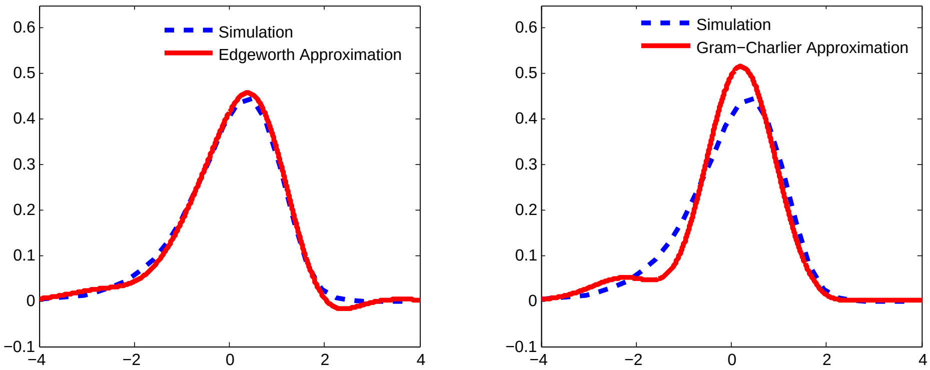

Figure 1: Simulated density vs analytical approximation for the Realized GARCH model. Left: Edgeworth, Right: Gram-Charlier

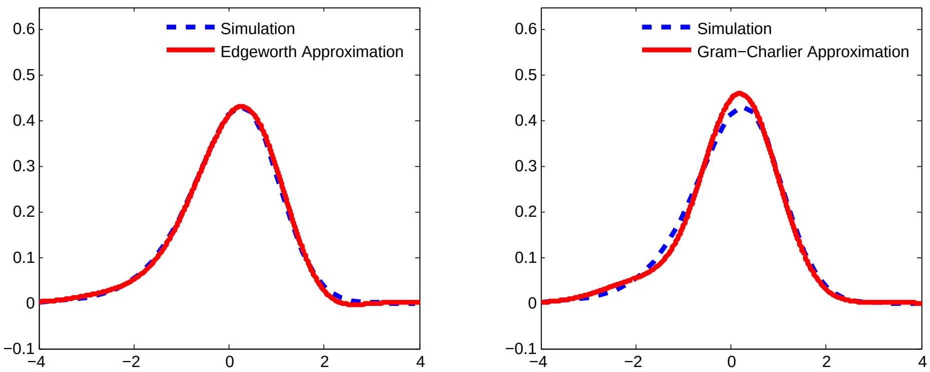

Figure 2: Simulated density vs analytical approximation for the EGARCH model. Left: Edgeworth, Right: Gram-Charlier

Table 1: Model Characteristics

Model	Parameters(P/Q)	Underlying	E Premium	V Premium	Spec.	Closed Fo
RG	12/10	Ret + RV	Yes	Yes	Log-linear	No
GARV	10/10	Ret + RV	Yes	Yes	Linear	Yes
EG	5/5	Ret	Yes	No	Log-linear	No
NG	5/4	Ret	Yes	No	Linear	No
GJR	5/5	Ret	Yes	No	Linear	No
HNG	5/4	Ret	Yes	No	Linear	Yes

E premium = Equity premium. V premium = Volatility premium. Spec. = Specification. Ret = Return. RV = Realized variance. Close Form = close form pricing formula. The parameter σ_e is not included in the count, because the weighted option pricing error.

Table 2: Option data set summary

Panel A: Wednesday: 2000-2012

		Maturity						
		<30	30−60	60−90	90−120	120−150	>150	Total
Delta	<0.3	369	549	548	434	240	225	2365
		(0.199)	(0.186)	(0.180)	(0.182)	(0.177)	(0.176)	(0.184)
	0.3−0.4	372	322	265	246	123	98	1426
		(0.178)	(0.183)	(0.186)	(0.200)	(0.191)	(0.194)	(0.187)
	0.4−0.5	406	443	382	261	109	122	1723
		(0.184)	(0.200)	(0.195)	(0.208)	(0.202)	(0.204)	(0.197)
	0.5−0.6	550	662	692	444	192	200	2740
		(0.192)	(0.200)	(0.209)	(0.227)	(0.222)	(0.217)	(0.208)
	0.6−0.7	514	579	545	417	214	217	2486
		(0.204)	(0.215)	(0.217)	(0.237)	(0.223)	(0.224)	(0.218)
	>0.7	989	1523	1502	1059	588	480	6141
		(0.249)	(0.251)	(0.252)	(0.263)	(0.244)	(0.243)	(0.252)
	Total	3200	4078	3934	2861	1466	1342	16881
		(0.210)	(0.218)	(0.219)	(0.231)	(0.219)	(0.218)	(0.219)

Panel B: Thursday: 2000-2012

		Maturity						
		<30	30−60	60−90	90−120	120−150	>150	Total
Delta	<0.3	270	535	563	410	249	213	2240
		(0.190)	(0.182)	(0.178)	(0.185)	(0.177)	(0.178)	(0.182)
	0.3−0.4	242	331	300	227	118	117	1335
		(0.169)	(0.184)	(0.186)	(0.205)	(0.189)	(0.189)	(0.186)
	0.4−0.5	297	420	379	256	114	106	1572
		(0.182)	(0.191)	(0.195)	(0.208)	(0.198)	(0.203)	(0.194)
	0.5−0.6	357	669	670	494	188	183	2561
		(0.188)	(0.200)	(0.204)	(0.221)	(0.219)	(0.214)	(0.206)
	0.6−0.7	342	606	516	377	207	211	2259
		(0.207)	(0.214)	(0.220)	(0.244)	(0.223)	(0.225)	(0.221)
	>0.7	750	1511	1456	1002	574	525	5818
		(0.248)	(0.253)	(0.250)	(0.263)	(0.245)	(0.244)	(0.252)
	Total	2258	4072	3884	2766	1450	1355	15785
		(0.208)	(0.217)	(0.217)	(0.231)	(0.219)	(0.219)	(0.219)

Panel C: Wednesday: 2013-2014

		Maturity						
		<30	30−60	60−90	90−120	120−150	>150	Total
Delta	<0.3	196	246	146	143	92	87	910
		(0.109)	(0.103)	(0.111)	(0.114)	(0.116)	(0.118)	(0.110)
	0.3−0.4	175	137	59	66	32	33	502
		(0.109)	(0.113)	(0.119)	(0.123)	(0.128)	(0.132)	(0.116)
	0.4−0.5	188	159	73	86	43	39	588
		(0.116)	(0.120)	(0.127)	(0.133)	(0.136)	(0.139)	(0.124)
	0.5−0.6	172	168	130	93	47	42	652
		(0.128)	(0.130)	(0.138)	(0.139)	(0.146)	(0.148)	(0.135)
	0.6−0.7	197	132	80	98	52	41	600
		(0.138)	(0.139)	(0.148)	(0.153)	(0.161)	(0.163)	(0.146)
	>0.7	503	656	410	401	201	164	2335
		(0.159)	(0.172)	(0.185)	(0.189)	(0.192)	(0.198)	(0.178)
	Total	1431	1498	898	887	467	406	5587
		(0.134)	(0.142)	(0.154)	(0.157)	(0.160)	(0.161)	(0.147)

Note: The number of options in each category is provided, average implied volatility is in parentheses.

Table 3: Parameter Estimation: 2000-2012

	Joint Estimation						Option NLS					
	RG	**GARV**	**EG**	**NG**	**GJR**	**HNG**	**RG**	**GARV**	**EG**	**NG**	**GJR**	**HNG**
λ	0.0056		0.0498	-0.0086	0.4319	2.1271	0.0680		0.0534		1.6790	
	(0.0021)		(0.0071)	(0.0016)	(0.0216)	(0.1794)	(0.0074)		(0.0087)		(0.0640)	
β	0.9896	0.9716	0.9847	0.8706	0.9247	0.6520	0.9876	0.9730	0.9848	0.8716	0.8841	0.9282
	(0.0003)	(0.0004)	(0.0005)	(0.0027)	(0.0009)	(0.0042)	(0.0004)	(0.0006)	(0.0005)	(0.0024)	(0.0023)	(0.0197)
τ_1	-0.1136	6.44E-07	-0.1138	0.0242	1.05E-08	1.39E-06	-0.1119	9.86E-07	-0.1142	0.0237	7.03E-05	1.93E-06
	(0.0013)	(8.88E-09)	(0.0011)	(0.0011)	(2.04E-09)	(1.98E-08)	(0.0016)	(2.59E-10)	(0.0012)	(0.0001)	(3.50E-06)	(1.66E-08)
τ_2	0.0052	203.7093	0.1072	2.0318	0.0716	493.3782	0.0271		0.1045	2.0419	0.0289	183.8286
	(0.0014)	(1.7195)	(0.0046)	(0.0831)	(0.0017)	(6.7182)	(0.0018)		(0.0047)	(0.0247)	(0.0011)	(26.8436)
γ	0.0479	0.3410					0.0270	0.1691				
	(0.0020)	(0.0036)					(0.0012)	(0.0003)				
ξ	-0.9318						-3.4029					
	(0.1079)						(0.2473)					
ϕ	0.9604	2.89E-06					0.6761	4.08E-06				
	(0.0124)	(4.90E-08)					(0.0277)	(5.32E-09)				
d_1	-0.3575	3.31E-06					-1.9987	1.73E-06				
	(0.0202)	(3.27E-09)					(0.0876)	(2.15E-11)				
d_2	0.1708	547.2332					0.7091					
	(0.0246)	(0.0040)					(0.0721)					
σ_u/ρ	0.6006	0.6145						1.0000				
	(0.0134)	(0.0033)						(0.0002)				
$\log h_1$	-8.9064	-7.8430	-8.5073	-8.0450	-9.7287	-8.5736	-9.0036	-7.8903	-8.5368	-8.1162	-10.8780	-9.2172
	(0.0443)	(0.0222)	(0.0549)	(0.0178)	(0.0375)	(0.0189)	(0.0595)	(0.0048)	(0.0673)	(0.0109)	(0.0379)	(0.0034)
χ	0.1438											
	(0.0150)											
κ		0.0795						0.0624				
		(0.0010)						(1.84E-05)				
τ_2^*		212.0384						157.0416				
		(1.4444)						(0.0108)				
d_2^*		547.2333						757.6637				
		(0.0601)						(0.5779)				
π^P	0.9896	0.9737	0.9847	0.9947	0.9604	0.9905						
π^Q	0.9896	0.9739	0.9847	0.9937	0.9926	0.9934	0.9876	0.9746	0.9848	0.9940	0.9942	0.9936
ℓ	44674.4	72128.1	39829.6	34156.9	43258.9	42319.5	39236.3	37498.3	29018.2	34403.1	34420.6	35859.0
$\sigma_e \times 100$	2.6318	2.6852	4.3373	3.1987	3.3326	3.5243	2.3678	2.6152	4.3347	3.1527	3.1466	2.8911

Note: Robust standard errors are in parenthesis. π^P and $\pi^Q$are persistence parameter under physical and risk neutral measure, respectively. ℓ is the log-likelihood value.

Table 4: In-sample pricing performance (IVRMSE): 2000-2012

	Joint Estimation						Option NLS					
	RG	**GARV**	**EG**	**NG**	**GJR**	**HNG**	**RG**	**GARV**	**EG**	**NG**	**GJR**	**HNG**
Total IVRMSE	2.7535	2.8340	4.1623	3.2943	3.3384	3.6223	2.4996	2.7619	4.1620	3.2140	3.1956	3.1630
Panel A: Partitioned by Moneyness												
Delta<0.3	3.2828	3.3007	6.3875	3.2396	3.6880	4.1020	3.1172	3.1257	6.4071	3.3584	3.3971	4.5787
0.3<Delta<0.4	2.6900	2.5604	3.9032	3.1451	3.4090	3.7455	2.3138	2.3599	3.9193	3.2057	3.2468	2.8676
0.4<Delta<0.5	2.4879	2.3969	3.0924	2.9750	3.2635	3.5579	2.1879	2.3097	3.1011	2.9748	3.0101	2.6825
0.5<Delta<0.6	2.4206	2.2658	3.0528	3.1384	3.2157	3.4156	2.1515	2.2870	3.0484	3.0267	3.0280	2.6401
0.6<Delta<0.7	2.4430	2.4154	3.4555	3.1900	3.2443	3.5298	2.2416	2.4637	3.4448	3.0475	3.0487	2.8771
0.7<Delta	2.8712	3.1789	4.0863	3.5350	3.2918	3.5458	2.5976	3.1051	4.0733	3.3668	3.2838	3.0063
Panel B: Partitioned by Maturity												
DTM<30	2.7995	3.2652	3.6549	3.5042	3.4462	4.3692	2.8230	3.2106	3.6564	3.5084	3.4394	3.5762
30<DTM<60	2.7073	2.9349	3.9897	3.3693	3.3927	3.8403	2.4469	2.9330	3.9885	3.3300	3.2938	3.1233
60<DTM<90	2.6282	2.6102	4.0865	3.1501	3.2853	3.3861	2.1723	2.5499	4.0870	3.0925	3.1077	2.8138
90<DTM<120	2.7189	2.5873	4.4875	3.2578	3.2740	3.1989	2.3158	2.5046	4.4853	3.0842	3.0292	3.0909
120<DTM<150	2.8367	2.7366	4.5691	3.0704	3.1064	3.1139	2.6177	2.6112	4.5684	2.9250	2.9566	3.0616
150<DTM	3.1000	2.6445	4.7973	3.2743	3.4436	2.9167	2.9403	2.3046	4.7979	3.0384	3.1397	3.4479
Panel C: Partitioned by VIX Level												
VIX<15	1.8738	1.6367	2.8550	2.5140	2.9901	3.3476	1.7898	1.9751	2.8707	2.9567	2.9889	2.2757
15<VIX<20	2.2592	1.6331	3.1342	2.1097	1.9572	2.8357	1.8649	1.6609	3.1325	1.9817	1.9247	2.8221
20<VIX<25	2.7668	2.6123	3.8954	3.4004	2.8277	3.5836	2.5934	2.4245	3.8990	3.1555	3.0235	2.8364
25<VIX<30	3.1139	3.4484	4.7295	3.9752	3.7164	3.3535	2.8019	3.1917	4.7284	3.7163	3.7071	3.0390
30<VIX<35	3.4683	4.5574	5.4179	4.5392	4.6994	3.2413	3.2488	4.5975	5.4102	4.2890	4.2004	3.2027
35<VIX	4.3327	5.2568	7.3792	5.2208	6.2940	6.3425	3.9796	4.9800	7.3656	5.1839	5.3807	5.8598

Table 5: Out-of-sample pricing performance (IVRMSE): Option NLS

	THU:2000-2012						WED:2013-2014					
	RG	**GARV**	**EG**	**NG**	**GJR**	**HNG**	**RG**	**GARV**	**EG**	**NG**	**GJR**	**HNG**
Total IVRMSE	2.5454	2.8563	4.4971	3.1438	3.1077	3.2677	1.7973	2.5931	2.9311	2.5458	2.6457	2.5740
Panel A: Partitioned by Moneyness												
Delta<0.3	3.0636	3.2299	5.2542	3.1570	3.1803	4.5078	1.6368	1.6738	4.5418	3.7498	3.8626	3.2896
0.3<Delta<0.4	2.2713	2.3885	3.4113	3.0300	3.0322	2.8185	1.4666	1.9671	3.1514	3.2297	3.3142	1.9304
0.4<Delta<0.5	2.2205	2.3949	2.6264	2.9109	2.8923	2.6944	1.4872	2.2178	2.4159	2.8452	2.9273	1.9486
0.5<Delta<0.6	2.1542	2.2826	2.4652	2.9360	2.9198	2.6804	1.4826	2.3725	1.8019	2.4086	2.5114	1.8369
0.6<Delta<0.7	2.3365	2.6451	3.2622	3.0775	3.0654	3.1101	1.5551	2.4447	1.2385	1.9474	2.0426	1.8163
0.7<Delta	2.6993	3.2041	5.7485	3.3331	3.2461	3.2317	2.1050	3.1319	2.7590	1.7755	1.8924	2.8440
Panel B: Partitioned by Maturity												
DTM<30	2.9313	3.5442	4.1466	3.3858	3.2490	3.9100	2.0460	1.8441	2.4664	2.4890	2.4926	1.9146
30<DTM<60	2.6189	3.1225	4.4584	3.2651	3.2108	3.3528	1.7165	2.0219	2.9540	2.6224	2.6665	1.9552
60<DTM<90	2.1623	2.5311	4.4763	3.0556	3.0471	2.8190	1.3719	2.6565	2.9855	2.4731	2.5980	2.4499
90<DTM<120	2.3374	2.5394	4.7283	3.0625	3.0077	3.1270	1.4127	3.1236	3.0120	2.4655	2.6448	3.1025
120<DTM<150	2.5744	2.6353	4.6437	2.8661	2.8994	3.0813	1.9063	3.4451	3.1933	2.6006	2.8233	3.5257
150<DTM	2.9834	2.4072	4.5922	3.0485	3.1399	3.4976	2.4749	3.8676	3.6506	2.7188	2.9686	3.9428
Panel C: Partitioned by VIX Level												
VIX<15	1.8025	2.0247	2.7714	2.8534	2.8963	2.3784	1.5366	2.7560	2.6552	2.6425	2.7472	2.6240
15<VIX<20	1.9174	1.7047	3.5871	2.0343	1.9729	2.7144	2.3417	2.0495	3.5494	2.2176	2.3144	2.4596
20<VIX<25	2.6013	2.6327	4.5338	3.2467	3.0915	2.9396	3.4225	1.5375	4.8413	2.3989	2.2825	1.5962
25<VIX<30	2.7868	3.0748	5.0020	3.3975	3.4072	2.8569						
30<VIX<35	2.9510	3.9653	5.6328	4.1517	3.9766	3.1246						
35<VIX	4.5712	5.8529	7.8506	5.1300	5.2706	6.9171						

Table 6: Out-of-sample pricing performance (IVRMSE): Joint estimation

	THU:2000-2012						WED:2013-2014					
	RG	**GARV**	**EG**	**NG**	**GJR**	**HNG**	**RG**	**GARV**	**EG**	**NG**	**GJR**	**HNG**
Total IVRMSE	2.7725	2.8965	4.1826	3.2298	3.2344	3.7043	1.6991	2.0950	2.9741	2.1879	2.9978	2.4326
Panel A: Partitioned by Moneyness												
Delta<0.3	3.2268	3.4104	6.3852	3.0294	3.4783	4.1636	1.7348	1.6608	5.5160	3.2627	4.3796	2.8654
0.3<Delta<0.4	2.6424	2.5526	3.8539	2.9746	3.1739	3.6039	1.5181	1.7719	3.6504	2.7638	3.7163	1.9490
0.4<Delta<0.5	2.5380	2.3994	3.0792	2.9262	3.0169	3.5634	1.4329	1.7861	2.8203	2.3890	3.2760	2.0372
0.5<Delta<0.6	2.4192	2.2165	2.9783	3.0502	3.1277	3.3764	1.3937	1.8233	2.2431	1.9659	2.9005	2.0369
0.6<Delta<0.7	2.4675	2.5195	3.5261	3.2314	3.2488	3.6818	1.3708	1.8145	1.4618	1.5583	2.3694	2.0193
0.7<Delta	2.9264	3.2588	4.1310	3.5038	3.2479	3.7288	1.9228	2.4849	1.5763	1.5884	2.1365	2.6265
Panel B: Partitioned by Maturity												
DTM<30	2.9104	3.4904	3.6495	3.3840	3.1222	4.6435	1.9151	1.7382	2.3658	2.2215	2.6605	1.8494
30<DTM<60	2.8092	3.0664	3.9827	3.3053	3.2580	4.0351	1.7849	1.7129	2.8039	2.3083	2.9283	2.0650
60<DTM<90	2.5874	2.5926	4.1125	3.1141	3.2461	3.4162	1.4706	2.1284	3.0168	2.1068	3.0242	2.2792
90<DTM<120	2.6701	2.5876	4.4694	3.2349	3.2680	3.2475	1.2889	2.4037	3.2274	2.0302	3.1346	2.8267
120<DTM<150	2.7439	2.8564	4.5772	3.0037	3.0693	3.1463	1.5289	2.5225	3.4936	2.1117	3.3559	3.3426
150<DTM	3.1493	2.7389	4.7184	3.2859	3.4113	3.0377	1.9863	2.9940	4.0076	2.2067	3.5265	3.4494
Panel C: Partitioned by VIX Level												
VIX<15	1.8944	1.6819	2.7633	2.4239	2.9270	3.1809	1.4702	2.2282	2.7437	2.2247	3.0981	2.4749
15<VIX<20	2.2955	1.6583	3.2308	2.1582	1.9724	2.8884	2.2444	1.6386	3.5773	2.0439	2.6956	2.3183
20<VIX<25	2.7564	2.8770	3.9467	3.4970	2.8493	3.6395	2.3894	1.4532	3.6374	2.4878	2.2777	1.9957
25<VIX<30	3.1679	3.3052	4.5462	3.6394	3.4376	3.3619						
30<VIX<35	3.0073	3.9852	5.0955	4.4413	4.3404	3.0218						
35<VIX	4.7662	5.8695	7.9590	5.1880	6.2461	7.3641						

Appendix A: Proof of Proposition 1

Lemma 1. *Let z follow the standard normal distribution. The integration*

$$G(k,\sigma,n) = \int_{-\infty}^{k} (z-\sigma)^n \phi(z)\, \mathrm{d}z$$

for any $n \in N$ and $\sigma \in R^+$ satisfies the following iteration equations

$$\begin{aligned}
&G(k,\sigma,n+2) = (n+1)\, G(k,\sigma,n) - \sigma G(k,\sigma,n+1) - (k-\sigma)^{n+1}\phi(k) \\
&G(k,\sigma,0) = \Phi(k) \\
&G(k,\sigma,1) = -\sigma\Phi(k) - \phi(k)
\end{aligned}$$

Proof. Integrating by part yields

$$\begin{aligned}
G(k,\sigma,n) &= \tfrac{1}{n+1}(k-\sigma)^{n+1}\phi(k) + \tfrac{1}{n+1}\int_{-\infty}^{k}(z-\sigma)^{n+2}\phi(z)\,\mathrm{d}z \\
&= +\tfrac{\sigma}{n+1}\int_{-\infty}^{k}(z-\sigma)^{n+1}\phi(z)\,\mathrm{d}z
\end{aligned}$$

Multiplying $(n+1)$ on both sides, we have

$$(n+1)\,G(k,\sigma,n) = (k-\sigma)^{n+1}\phi(k) + G(k,\sigma,n+2) + \sigma G(k,\sigma,n+1)$$

Which can be rearranged as

$$G(k,\sigma,n+2) = (n+1)\,G(k,\sigma,n) - \sigma G(k,\sigma,n+1) - (k-\sigma)^{n+1}\phi(k)$$

It is easy to see that $G(k,\sigma,0) = \Phi(k)$. For $n = 1$,

$$G(k,\sigma,1) = \int_{-\infty}^{k}(z-\sigma)\phi(z)\,\mathrm{d}z = \int_{-\infty}^{k} z\phi(z)\,\mathrm{d}z - \sigma\Phi(k) = -\phi(k) - \sigma\Phi(k)$$

□

Proof outline of Proposition 1

Proof. By definition, we have $S_T = S_0 \exp(R_T) = S_0 \exp(\mu + \sigma z)$ so that

$$S_T \geq K \Leftrightarrow -z \leq \tfrac{\log(S_0/K)+\mu}{\sigma}.$$

So if we set $k = \{\log(S_0/K) + \mu\}/\sigma$ the price of an European call is then:

$$e^{-rT}\mathbb{E}_0^Q(\max(S_T - K, 0)) = e^{-rT}\int_{-\infty}^{k}[S_0 \exp(\mu - \sigma z) - K]\, g(z)\,\mathrm{d}z,$$

where g is the density of z, We approximate this density using the following 2nd order Edgeworth expansion with a Gaussian reference density:

$$\tilde{g}(z) = \left\{1 - \tfrac{k_3}{6}H_3(z) + \tfrac{(k_4-3)}{24}H_4(z) + \tfrac{k_3^2}{72}H_6(z)\right\}\phi(z),$$

where $H_3(z) = z^3 - 3z$, $H_4(z) = z^4 - 6z^2 + 3$, and $H_6(z) = z^6 - 15z^4 + 45z^2 - 15$ are Hermite polynomials. With this analytical approximation we have:

$$\begin{aligned} e^{-rT}\int_{-\infty}^{k}(S_T - K)\tilde{g}(z)\,\mathrm{d}z &= e^{-rT}\int_{-\infty}^{k}\left[S_0 e^{\mu-\sigma z} - K\right]\phi(z)\,\mathrm{d}z \quad (A) \\ &\quad -\tfrac{k_3}{6}e^{-rT}\int_{-\infty}^{k}\left[S_0 e^{\mu-\sigma z} - K\right]H_3(z)\phi(z)\,\mathrm{d}z \quad (B) \\ &\quad +\tfrac{(k_4-3)}{24}e^{-rT}\int_{-\infty}^{k}\left[S_0 e^{\mu-\sigma z} - K\right]H_4(z)\phi(z)\,\mathrm{d}z \quad (C) \\ &\quad +\tfrac{k_3^2}{72}e^{-rT}\int_{-\infty}^{k}\left[S_0 e^{\mu-\sigma z} - K\right]H_6(z)\phi(z)\,\mathrm{d}z. \quad (D) \end{aligned}$$

The first three terms, A, B, and C, are derived in Duan et al. (1999), and the fourth and last term, D, is derived next.

$$\begin{aligned} & e^{-rT}\int_{-\infty}^{k}\left[S_0 e^{\mu-\sigma z} - K\right]H_6(z)\phi(z)\,\mathrm{d}z \\ = \; & \underbrace{S_0\, e^{-rT}\int_{-\infty}^{k} e^{\mu-\sigma z}H_6(z)\phi(z)\mathrm{d}z}_{=D_1} \\ & -Ke^{-rT}\underbrace{\int_{-\infty}^{k} H_6(z)\phi(z)\mathrm{d}z}_{=D_2} \end{aligned}$$

For D_1 we have

$$\begin{aligned} D_1 &= e^{-rT}\int_{-\infty}^{k}\frac{1}{\sqrt{2\pi}}\exp\left(-\tfrac{z^2}{2} - \sigma z + \mu\right)\left(z^6 - 15z^4 + 45z^2 - 15\right)\mathrm{d}z \\ &= e^{-rT+\frac{\sigma^2}{2}+\mu}\int_{-\infty}^{k}\frac{1}{\sqrt{2\pi}}\exp\left(-\tfrac{(z+\sigma)^2}{2}\right)\left(z^6 - 15z^4 + 45z^2 - 15\right)\mathrm{d}z \\ &= e^{\delta\sigma}\int_{-\infty}^{d}\left((x-\sigma)^6 - 15(x-\sigma)^4 + 45(x-\sigma)^2 - 15\right)\phi(x)\mathrm{d}x, \end{aligned}$$

where δ, d are given in Proposition 1. From Lemma 1, we have

$$\begin{aligned} D_1 &= e^{\delta\sigma}\left(G(d,\sigma,6) - 15G(d,\sigma,4) + 45G(d,\sigma,2) - 15\right) \\ D_2 &= G(k,0,6) - 15G(k,0,4) + 45G(k,0,2) - 15. \end{aligned}$$

Collecting terms and simple algebra reveals that D is equal to:

$$\frac{k_3^2}{72}S_0 e^{\delta\sigma}\sigma\left[\sigma^5\Phi(d) + \left(3 - 6d^2 + d^4 + 5\sigma\left(d - (d-\sigma)(\sigma d - 2) - (d-\sigma)^3\right)\right)\phi(d)\right].$$

□

Appendix B: Analytical results for the included terms

The moment of cumulative returns can be expressed as:

$$\mathbb{E}_0^Q(R_T^s) = \mathbb{E}_0^Q\left[\sum_{i=1}^{T}\left(r - \tfrac{1}{2}h_{t+i} + \sqrt{h_{t+i}}z_{t+i}\right)^s\right]$$

Expanding the formula, we have:

$$\begin{aligned}
\mathbb{E}_T^Q\left(R_T\right) &= Tr - \tfrac{1}{2}\sum_{i=1}^{T}\mathbb{E}_0^Q\left[h_i\right] \\
\mathbb{E}_0^Q\left(R_T^2\right) &= T^2r^2 - Tr\sum_{i=1}^{T}\mathbb{E}_0^Q\left[h_i\right] + \frac{1}{4}S_{D1} + S_{D2} - S_{D3} \\
\mathbb{E}_0^Q\left(R_T^3\right) &= T^3r^3 - \frac{3}{2}T^2r^2\sum_{i=1}^{T}\mathbb{E}_0^Q\left[h_i\right] + 3Tr\left(\frac{1}{4}S_{D1} + S_{D2} - S_{D3}\right) \\
&\quad + \left(-\frac{1}{8}S_{T1} + S_{T2} + \frac{3}{4}S_{T3} - \frac{3}{2}S_{T4}\right) \\
\mathbb{E}_0^Q\left(R_T^4\right) &= T^4r^4 - 2T^3r^3\sum_{i=1}^{T}\mathbb{E}_0^Q\left[h_i\right] + 6T^2r^2\left(\frac{1}{4}S_{D1} + S_{D2} - S_{D3}\right) \\
&\quad + Tr\left(-\tfrac{1}{2}S_{T1} + 4S_{T2} + 3S_{T3} - 6S_{T4}\right) \\
&\quad + \left(\frac{1}{16}S_{Q1} + S_{Q2} - \tfrac{1}{2}S_{Q3} + \frac{3}{2}S_{Q4} - 2S_{Q5}\right).
\end{aligned}$$

We use S_{D3} and S_{T1} as examples to illustrate how those terms are related with summations of expectations of future volatility and shocks. Readers are encouraged to see Duan et al. (1999) for detailed information of other S_{Di}, S_{Ti} and S_{Qi}.

$$S_{D3} = \mathbb{E}_0^Q\left[\sum_{i=1}^{T}\sum_{j=1}^{T}h_i\sqrt{h_j}z_j\right] = \sum_{i=1}^{T}\sum_{j=1}^{T-i}\mathbb{E}_0^Q[\sqrt{h_i}z_ih_{i+j}].$$

So in order to compute S_{D3} we need to calculate (B.6) (defined below).

$$\begin{aligned}
S_{T1} &= \mathbb{E}_0^Q\left[\sum_{i=1}^{T}\sum_{j=1}^{T}\sum_{k=1}^{T}h_ih_jh_k\right] \\
&= 6\sum_{i=1}^{T}\sum_{j=1}^{T-i}\sum_{k=1}^{T-i-j}\mathbb{E}_0^Q[h_ih_{i+j}h_{i+j+k}] + 3\sum_{i=1}^{T}\sum_{j=1}^{T-i}\mathbb{E}_0^Q[h_i^2h_{i+j}] \\
&\quad + 3\sum_{i=1}^{T}\sum_{j=1}^{T-i}\mathbb{E}_0^Q[h_ih_{i+j}^2] + \sum_{i=1}^{T}\mathbb{E}_0^Q[h_i^3].
\end{aligned}$$

Similarly, to compute S_{T3} we need the terms, (B.3), (B.4), (B.5) and (B.1) for $m = 3$, that are defined below.

Terms Needed for S_{Di}, S_{Ti} and S_{Qi}

In this section we derive the key terms that are needed to evaluate S_{Di}, S_{Ti} and S_{Qi}. From the risk neutral dynamics, we simplify the notation as

$$\log h_{t+1} = \tilde{\omega} + \beta \log h_t + v_t,$$

where $\tilde{\omega} = \omega + \chi\tilde{\sigma}$, $\tilde{\sigma} = \gamma\sigma$, $v_t = \tau_t(z_t^* - \lambda) + \tau_2((z_t^* - \lambda)^2 - 1) + \tilde{\sigma}u_t^*$.

Expectations without z

Using the simplified notation, we have:

$$\begin{aligned}
\mathbb{E}_0^Q(h_i^m) &= \mathbb{E}_0^Q\left[\exp\left(m\beta^{i-1}\log h_{t+1}\right)\exp\left(\sum_{k=1}^{i-1} m\tilde{\omega}\beta^k\right)\exp\left(\sum_{k=1}^{i-1} m\beta^{i-1-k}v_{1+k}\right)\right] \\
&= h_1^{m\beta^{i-1}}\prod_{k=0}^{i-2} e^{m\beta^k\tilde{\omega}}\mathbb{E}_t^Q\left[e^{m\beta^k v_{1+k}}\right].
\end{aligned}$$

Let $F_k(m) = e^{m\beta^k\tilde{\omega}}\mathbb{E}_0^Q\left[e^{m\beta^k v_{1+k}}\right]$, suppress star and t on z and u. We have

$$\begin{aligned}
F_k(m) &= e^{m\beta^k\tilde{\omega}}\mathbb{E}_0^Q\left[\exp\left(m\beta^k\tau_2\lambda^2 - m\beta^k\tau_1\lambda - m\beta^k\tau_2 + m\beta^k\left(\tau_2 z^2 - (2\tau_2\lambda - \tau_1)z\right)\right)\exp\left(m\beta^k\tilde{\sigma}u\right)\right] \\
&= e^{m\beta^k\tilde{\omega}}\exp\left[m\beta^k\tau_2\lambda^2 - m\beta^k\tau_1\lambda - m\beta^k\tau_2 + \tfrac{m^2\beta^{2k}\tilde{\sigma}^2}{2}\right]\mathbb{E}_t^Q\exp\left(m\beta^k\left(\tau_2 z^2 - (2\tau_2\lambda - \tau_1)z\right)\right) \\
&= e^{m\beta^k\tilde{\omega}}\exp\left[m\beta^k\left(\tau_2(\lambda^2-1) - \tau_1\lambda + \tfrac{m\beta^k\tilde{\sigma}^2}{2} - \tfrac{(\tau_1 - 2\tau_2\lambda)^2}{4\tau_2}\right)\right] \\
&\quad \times\mathbb{E}_t^Q\left[m\beta^k\tau_2\left(z + \tfrac{(\tau_1 - 2\tau_2\lambda)}{2\tau_2}\right)^2\right].
\end{aligned}$$

The last term in the third equation is essentially a moment-generating function of the non-central chi square distribution. Therefore we have:

$$\mathbb{E}_0^Q\left[m\beta^k\tau_2\left(z + \tfrac{(\tau_1 - 2\tau_2\lambda)}{2\tau_2}\right)^2\right] = \frac{1}{\sqrt{1-2m\beta^k\tau_2}}\exp\left[\frac{m\beta^k(\tau_1 - 2\tau_2\lambda)^2}{4(1-2m\beta^k\tau_2)\tau_2}\right].$$

Next, we substitute the expression into $F_k(m)$, and find

$$F_k(m) = \frac{1}{\sqrt{1-2m\beta^k\tau_2}}\exp\left[m\beta^k\left(\tilde{\omega} + \tau_2(\lambda^2-1) - \tau_1\lambda + \tfrac{m\beta^k\tilde{\sigma}^2}{2} + \tfrac{m\beta^k(\tau_1 - 2\tau_2\lambda)^2}{2(1-2m\beta^k\tau_2)}\right)\right].$$

Therefore we have:

$$\mathbb{E}_0^Q(h_i^m) = h_1^{m\beta^{i-1}}\prod_{k=1}^{i-2} F_k(m) \qquad (B.1)$$

. Similar techniques yield:

$$\mathbb{E}_0^Q(h_i h_{i+j}) = \mathbb{E}_0^Q[h_i^{\beta^j+1}]\prod_{k=1}^{j} F_k(1) = \mathbb{E}_0^Q[h_i^{\beta^j+1}]\mathbb{E}_0^Q[h_{j+1}]h_1^{-\beta^j} \qquad (B.2)$$

$$\mathbb{E}_0^Q(h_i h_{i+j} h_{i+j+k}) = \mathbb{E}_0^Q\left[h_i^{\beta^j+1+\beta^{j+k}}\right]\mathbb{E}_0^Q\left[h_{j+1}^{1+\beta^k}\right]\mathbb{E}_0^Q\left[h_{k+1}\right]h_1^{-\beta^j(1+\beta^k)-\beta^k} \qquad (B.3)$$

$$\mathbb{E}_0^Q(h_i^2 h_{i+j}) = \mathbb{E}_0^Q\left[h_i^{\beta^j+2}\right]\mathbb{E}_0^Q\left[h_{j+1}\right]h_1^{-\beta^j}, \qquad (B.4)$$

$$\mathbb{E}_0^Q(h_i h_{i+j}^2) = \mathbb{E}_0^Q\left[h_i^{2\beta^j+1}\right]\mathbb{E}_0^Q\left[h_{j+1}^2\right]h_1^{-2\beta^j}. \qquad (B.5)$$

Expectations with z

When z is involved, the following auxiliary results are needed. Let $Y_i = \tilde{\omega} + v_i$, then we have:

$$
\begin{aligned}
\mathbb{E}_0^Q(z_i \exp(kY_i)) &= \mathbb{E}_0^Q\left[z_i \exp\left(k\tilde{\omega} + k\tau_1(z_i-\lambda) + k\tau_2[(z_i-\lambda)^2-1] + k\tilde{\sigma}u_i\right)\right] \\
&= \mathbb{E}_0^Q\left[z_i \exp\left(k\tau_2 z_i^2 + k(\tau_1 - 2\tau_2\lambda)z_i\right)\right] \exp\left(k\tilde{\omega} - k\tau_1\lambda + k\tau_2(\lambda^2-1) + \tfrac{k^2\tilde{\sigma}^2}{2}\right) \\
&= \exp\left(k\tilde{\omega} - k\tau_1\lambda + k\tau_2(\lambda^2-1) + \tfrac{b^2}{4a} + \tfrac{k^2\tilde{\sigma}^2}{2}\right) \tfrac{1}{\sqrt{2a}} \int z \tfrac{1}{\sqrt{2\pi/2a}} \exp(-\tfrac{(z-b/2a)^2}{2/a}) \mathrm{d}z \\
&= \exp\left(k\tilde{\omega} - k\tau_1\lambda + k\tau_2(\lambda^2-1) + \tfrac{b^2}{4a} + \tfrac{k^2\tilde{\sigma}^2}{2}\right) \tfrac{b}{(2a)^{3/2}}, \\
\mathbb{E}_0^Q(z_i^2 \exp(kY_i)) &= \mathbb{E}_0^Q\left[z_i^2 \exp\left(k\tilde{\omega} + k\tau_1(z_i-\lambda) + k\tau_2[(z_i-\lambda)^2-1] + k\tilde{\sigma}u_i\right)\right] \\
&= \mathbb{E}_0^Q\left[z_i^2 \exp\left(k\tau_2 z_i^2 + k(\tau_1 - 2\tau_2\lambda)z_i\right)\right] \exp\left(k\tilde{\omega} - k\tau_1\lambda + k\tau_2(\lambda^2-1) + \tfrac{k^2\tilde{\sigma}^2}{2}\right) \\
&= \exp\left(k\tilde{\omega} - k\tau_1\lambda + k\tau_2(\lambda^2-1) + \tfrac{b^2}{4a} + \tfrac{k^2\tilde{\sigma}^2}{2}\right) \tfrac{1}{\sqrt{2a}} \int z^2 \tfrac{1}{\sqrt{2\pi/2a}} \exp(-\tfrac{(z-b/2a)^2}{2/2a}) \mathrm{d}z \\
&= \exp\left(k\tilde{\omega} - k\tau_1\lambda + k\tau_2(\lambda^2-1) + \tfrac{b^2}{4a} + \tfrac{k^2\tilde{\sigma}^2}{2}\right) \tfrac{b^2+2a}{(2a)^{5/2}}, \\
\mathbb{E}_0^Q(z_i^3 \exp(kY_i)) &= \mathbb{E}_0^Q\left[z_i^3 \exp\left(k\tilde{\omega} + k\tau_1(z_i-\lambda) + k\tau_2[(z_i-\lambda)^2-1] + k\tilde{\sigma}u_i\right)\right] \\
&= \mathbb{E}_0^Q\left[z_i^3 \exp\left(k\tau_2 z_i^2 + k(\tau_1 - 2\tau_2\lambda)z_i\right)\right] \exp\left(k\tilde{\omega} - k\tau_1\lambda + k\tau_2(\lambda^2-1) + \tfrac{k^2\tilde{\sigma}^2}{2}\right) \\
&= \exp\left(k\tilde{\omega} - k\tau_1\lambda + k\tau_2(\lambda^2-1) + \tfrac{b^2}{4a} + \tfrac{k^2\tilde{\sigma}^2}{2}\right) \tfrac{1}{\sqrt{2a}} \int z^3 \tfrac{1}{\sqrt{2\pi/2a}} \exp(-\tfrac{(z-b/2a)^2}{2/2a}) \mathrm{d}z \\
&= \exp\left(k\tilde{\omega} - k\tau_1\lambda + k\tau_2(\lambda^2-1) + \tfrac{b^2}{4a} + \tfrac{k^2\tilde{\sigma}^2}{2}\right) \tfrac{b^3+6ab}{(2a)^{7/2}},
\end{aligned}
$$

where $a = \frac{1}{2} - k\tau_2$,$b = k(\tau_1 - 2\tau_2\lambda)$. In the following calculations, we still use the similar technique in deriving (B.1) and link expectations with these terms.

$$\mathbb{E}_0^Q(\sqrt{h_i} z_i h_{i+j}) = \mathbb{E}_0^Q\left[h_i^{(\beta^j+\frac{1}{2})}\right] \mathbb{E}_0^Q\left[z_i \exp(\beta^{j-1}Y_i)\right] \mathbb{E}_0^Q\left[h_j\right] h_i^{-\beta^{j-1}} \tag{B.6}$$

$$
\begin{aligned}
\mathbb{E}_0^Q(h_i \sqrt{h_{i+j}} z_{i+j} h_{i+j+k}) &= \mathbb{E}_0^Q\left[z_{i+j} \exp\left(\beta^{k-1}Y_{i+j}\right)\right] \mathbb{E}_0^Q\left[h_i^{\beta^j(\beta^k+\frac{1}{2})+1}\right] \mathbb{E}_0^Q\left[h_{j+1}^{(\beta^k+\frac{1}{2})}\right] \\
&\quad \times \mathbb{E}_0^Q\left(h_k\right) h_1^{-\beta^j(\beta^k+\frac{1}{2})-\beta^{k-1}}
\end{aligned} \tag{B.7}
$$

$$
\begin{aligned}
\mathbb{E}_0^Q(\sqrt{h_i} z_i h_{i+j} h_{i+j+k}) &= \mathbb{E}_0^Q\left[z_i \exp\left(\beta^{j-1}(1+\beta^k)Y_i\right)\right] \mathbb{E}_0^Q\left[h_i^{\beta^j(\beta^k+1)+\frac{1}{2}}\right] \mathbb{E}_0^Q\left[h_j^{(\beta^k+1)}\right] \\
&\quad \times \mathbb{E}_0^Q\left(h_{k+1}\right) h_1^{-\beta^{j-1}(\beta^k+1)-\beta^k}
\end{aligned} \tag{B.8}
$$

$$\mathbb{E}_0^Q(h_i^{3/2} z_i h_{i+j}) = \mathbb{E}_0^Q\left[z_i \exp\left(\beta^{j-1}Y_i\right)\right] \mathbb{E}_0^Q\left[h_i^{\beta^j+\frac{3}{2}}\right] \mathbb{E}_0^Q\left[h_j\right] h_1^{-\beta^{j-1}} \tag{B.9}$$

$$\mathbb{E}_0^Q(\sqrt{h_i} z_i h_{i+j}^2) = \mathbb{E}_0^Q\left[z_i \exp\left(2\beta^{j-1}Y_i\right)\right] \mathbb{E}_0^Q\left[h_i^{2\beta^j+\frac{1}{2}}\right] \mathbb{E}_0^Q\left[h_j^2\right] h_1^{-2\beta^{j-1}} \tag{B.10}$$

$$
\begin{aligned}
\mathbb{E}_0^Q(\sqrt{h_i} z_i \sqrt{h_{i+j}} z_{i+j} h_{i+j+k}) &= \mathbb{E}_0^Q\left[z_i \exp\left(\beta^{j-1}\left(\tfrac{1}{2}+\beta^k\right)Y_i\right)\right] \mathbb{E}_0^Q\left[z_{i+j} \exp\left(\beta^{k-1}Y_{i+j}\right)\right] \\
&\quad \times \mathbb{E}_0^Q\left[h_i^{\beta^j\left(\beta^k+\frac{1}{2}\right)+\frac{1}{2}}\right] \mathbb{E}_0^Q\left[h_j^{\left(\beta^k+\frac{1}{2}\right)}\right] \\
&\quad \times \mathbb{E}_0^Q[h_k] h_1^{-\beta^{j-1}(\beta^k+\frac{1}{2})-\beta^{k-1}}
\end{aligned} \tag{B.11}
$$

$$\mathbb{E}_0^Q(h_i z_i^2 h_{i+j}) = \mathbb{E}_0^Q\left[z_i^2 \exp\left(\beta^{j-1}Y_i\right)\right] \mathbb{E}_0^Q\left[h_i^{\beta^j+1}\right] \mathbb{E}_0^Q\left[h_j\right] h_1^{-\beta^{j-1}} \tag{B.12}$$

With $a = \beta^m + 1$ and $b = a\beta^k + \frac{1}{2}$:

$$
\begin{aligned}
&\mathbb{E}_0^Q(\sqrt{h_i}z_i\sqrt{h_{i+j}}z_{i+j}h_{i+j+k}h_{i+j+k+m}) \\
&\quad= \mathbb{E}_0^Q\left(\sqrt{h_i}z_i\sqrt{h_{i+j}}z_{i+j}h_{i+j+k}h_{i+j+k}^{\beta^m}\exp\left(\sum_{w=1}^{k}\beta^{w-1}Y_{i+j+k+m-w}\right)\right) \\
&\quad= \mathbb{E}_0^Q\left(\sqrt{h_i}z_i\sqrt{h_{i+j}}z_{i+j}h_{i+j+k}^{a}\left[h_1^{\beta^m}\exp\left(\sum_{w=1}^{k}\beta^{w-1}Y_{i+j+k+m-w}\right)\right]\right)h_1^{-\beta^m} \\
&\quad= \mathbb{E}_0^Q\left(\sqrt{h_i}z_i\sqrt{h_{i+j}}z_{i+j}h_{i+j+k}^{a}\right)\mathbb{E}_0^Q(h_{m+1})h_1^{-\beta^m} \\
&\quad= \mathbb{E}_0^Q\left(\sqrt{h_i}z_ih_{i+j}^{b}\right)\mathbb{E}_0^Q\left(z_{i+j}\exp\left(a\beta^{k-1}Y_{i+j}\right)\right)\mathbb{E}_0^Q(h_k^a)\mathbb{E}_0^Q(h_{m+1})h_1^{-\beta^m}h_1^{-a\beta^{k-1}} \\
&\quad= \mathbb{E}_0^Q\left(h_i^{b\beta^j+\frac{1}{2}}\right)\mathbb{E}_0^Q(h_j^b)\mathbb{E}_0^Q(h_k^a)\mathbb{E}_0^Q(h_{m+1})\mathbb{E}_0^Q\left(z_i\exp\left(b\beta^{j-1}Y_i\right)\right) \\
&\qquad\times\mathbb{E}_0^Q\left(z_{i+j}\exp\left(a\beta^{k-1}Y_{i+j}\right)\right)h_1^{-(\beta^m+a\beta^{k-1}+b\beta^{j-1})}
\end{aligned}
\tag{B.13}
$$

With $m = \beta^k + 1$

$$
\mathbb{E}_0^Q(h_iz_i^2h_{i+j}h_{i+j+k}) = \mathbb{E}_0^Q\left[h_i^{m\beta^j+1}\right]\mathbb{E}_0^Q\left[h_j^m\right]\mathbb{E}_0^Q\left[h_{k+1}\right]\times\mathbb{E}_0^Q\left[z_i^2\exp\left(\beta^{j-1}Y_i\right)\right]h_1^{-(\beta^k+m\beta^{j-1})}
\tag{$B.14$}
$$

$$
\begin{aligned}
\mathbb{E}_0^Q(h_ih_{i+j}z_{i+j}^2h_{i+j+k}) &= \mathbb{E}_0^Q\left(h_ih_{i+j}z_{i+j}^2h_{i+j}^{\beta^k}\exp\left(\sum_{w=1}^{k}\beta^{w-1}Y_{i+j+k-w}\right)\right) \\
&= \mathbb{E}_0^Q\left(h_ih_{i+j}^{\beta^k+1}z_{i+j}^2\exp\left(\beta^{k-1}Y_{i+j}\right)h_1^{\beta^{k-1}}\exp\left(\sum_{w=1}^{k-1}\beta^{w-1}Y_{i+j+k-w}\right)\right)h_1^{-\beta^{k-1}} \\
&= \mathbb{E}_0^Q\left(h_ih_{i+j}^{\beta^k+1}\right)\mathbb{E}_0^Q\left(z_{i+j}^2\exp\left(\beta^{k-1}Y_{i+j}\right)\right)\mathbb{E}_0^Q(h_k)h_1^{-\beta^{k-1}} \\
&= \mathbb{E}_0^Q\left[h_i^{m\beta^j+1}\right]\mathbb{E}_0^Q\left[h_{j+1}^m\right]\mathbb{E}_0^Q\left[h_k\right]\mathbb{E}_0^Q\left[z_{i+j}^2\exp\left(\beta^{k-1}Y_{i+j}\right)\right]h_1^{-(\beta^{k-1}+m\beta^j)}
\end{aligned}
\tag{B.15}
$$

$$
\begin{aligned}
\mathbb{E}_0^Q(\sqrt{h_i}z_ih_{i+j}z_{i+j}^2h_{i+j+k}) &= \mathbb{E}_0^Q\left[h_i^{m\beta^j+\frac{1}{2}}\right]\mathbb{E}_0^Q\left[h_j^m\right]\mathbb{E}_0^Q\left[h_k\right]\mathbb{E}_0^Q\left[z_i\exp\left(m\beta^{j-1}Y_i\right)\right]\times \\
&\quad \mathbb{E}_0^Q\left[z_{i+j}^2\exp\left(\beta^{k-1}Y_{i+j}\right)\right]h_1^{-(\beta^{k-1}+m\beta^{j-1})}
\end{aligned}
\tag{B.16}
$$

With $w = \beta^m + \frac{1}{2}$ and $s = w\beta^k + \frac{1}{2}$:

$$
\begin{aligned}
&\mathbb{E}_0^Q(\sqrt{h_i}z_i\sqrt{h_{i+j}}z_{i+j}\sqrt{h_{i+j+k}}z_{i+j+k}h_{i+j+k+m}) \\
&= \mathbb{E}_0^Q\left(h_i^{s\beta^j+\frac{1}{2}}\right)\mathbb{E}_0^Q\left(z_i\exp\left(s\beta^{j-1}Y_i\right)\right)\mathbb{E}_0^Q\left(z_{i+j}\exp\left(w\beta^{k-1}Y_{i+j}\right)\right)\mathbb{E}_0^Q\left(z_{i+j+k}\exp\left(\beta^{m-1}Y_{i+j+k}\right)\right) \\
&\quad\times\mathbb{E}_0^Q(h_j^s)\mathbb{E}_0^Q(h_k^w)\mathbb{E}_0^Q(h_m)h_1^{-(\beta^m+w\beta^{k-1}+s\beta^{j-1})}
\end{aligned}
\tag{B.17}
$$

With $\tilde{m} = \beta^k + \frac{1}{2}$:

$$
\begin{aligned}
\mathbb{E}_0^Q(h_iz_i^2\sqrt{h_{i+j}}z_{i+j}h_{i+j+k}) &= \mathbb{E}_0^Q\left[h_i^{\tilde{m}\beta^j+1}\right]\mathbb{E}_0^Q\left[h_j^{\tilde{m}}\right]\mathbb{E}_0^Q\left[h_k\right]\mathbb{E}_0^Q\left[z_i^2\exp\left(\tilde{m}\beta^{j-1}Y_i\right)\right]\times \\
&\quad \mathbb{E}_0^Q\left[z_{i+j}\exp\left(\beta^{k-1}Y_{i+j}\right)\right]h_1^{-(\beta^{k-1}+\tilde{m}\beta^{j-1})}
\end{aligned}
\tag{B.18}
$$

$$
\mathbb{E}_0^Q(h_i^{3/2}z_i^3h_{i+j}) = \mathbb{E}_0^Q\left[z_i^3\exp\left(\beta^{j-1}Y_i\right)\right]\mathbb{E}_0^Q\left[h_i^{\beta^j+\frac{3}{2}}\right]\mathbb{E}_0^Q\left[h_j\right]h_1^{-\beta^{j-1}}
\tag{$B.19$}
$$

Table B.1 provides a summary of here these 19 formulas are used. Duan et al. (2006) omitted several "small" terms especially in S_{Qi} and we follow their approach in this paper.

Table B.1: Formulas used in evaluation S_{Di}, S_{Ti} and S_{Qi}

	Formula number B.x							
S_{D1}	1	2						
S_{D2}	1							
S_{D3}	6							
S_{T1}	1	3	4	5				
S_{T2}	6							
S_{T3}	7	8	9	10				
S_{T4}	1	2	11	12				
S_{Q2}	1	11	12					
S_{Q4}	4	6	13	14	15			
S_{Q5}	7	8	9	10	16	17	18	19

Finally, the expectation of fractional powered h is evaluated through Taylor expansion:

$$\begin{aligned}
\mathbb{E}_0^Q\left[h_t^a\right] \approx\ & \left(1+\tfrac{32}{12}a-\tfrac{23}{8}a^2+\tfrac{13}{12}a^3-\tfrac{1}{8}a^4\right)\mathbb{E}_0^Q\left[h_t\right]^a \\
& +\left(-3a+\tfrac{19}{4}a^2-2a^3+\tfrac{1}{4}a^4\right)\mathbb{E}_0^Q\left[h_t\right]^{a-2}\mathbb{E}_0^Q\left[h_t^2\right] \\
& +\left(\tfrac{4}{3}a-\tfrac{7}{3}a^2+\tfrac{7}{6}a^3-\tfrac{1}{6}a^4\right)\mathbb{E}_0^Q\left[h_t\right]^{a-3}\mathbb{E}_0^Q\left[h_t^3\right] \\
& +\left(-\tfrac{1}{4}a+\tfrac{11}{24}a^2-\tfrac{1}{4}a^3+\tfrac{1}{24}a^4\right)\mathbb{E}_0^Q\left[h_t\right]^{a-4}\mathbb{E}_0^Q\left[h_t^4\right].
\end{aligned}$$

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