Endowment Structure, Industrialization and Post-industrialization: A Three-Sector Model of Structural Change

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Abstract

The paper aims to provide a unified framework to characterize both the stage of industrialization and of post-industrialization during the history of economic development. The updating of endowment structure tends to increase the relative output of the sectors with higher capital intensity, and then the gross complementarity between the labor-intensive service goods and the capital-intensive compound substitute goods between agricultural goods and industrial goods (industrial goods is more capital-intensive than agricultural goods), would initially lead to the flow of factors from agriculture to industry, and later from agriculture and industry to services. In addition, our model establishes the consistency of sectoral change in allocation of factors with the asymptotically constant interest rate and aggregate growth rate in the setup with three sectors. Our model can be viewed as a natural extension of the work of Acemoglu and Guerriieri (2006) in the sense that it introduces more general setup and thus has potential to explain more interesting phenomenon in the sectoral change.

Keywords: endowment structure, industrialization and post-industrialization, structural change.

I. Introduction

Recently, economists researching structural change have paid more attention to the puzzle whether "Kuznets facts", the systematic change in relative importance between different sectors, in particular, agriculture, manufacturing and services, can be consistent with "Kaldors facts", the relative constancy of the real interest, the aggregate growth rate, the capital-output ratio and the share of labor or capital income in GDP. The latest research on this theme are three independent papers by Kongsamut, Rebelo and Xie (2001), Ngai and Pissarides (2006) and Acemoglu and Guerriieri (2006).

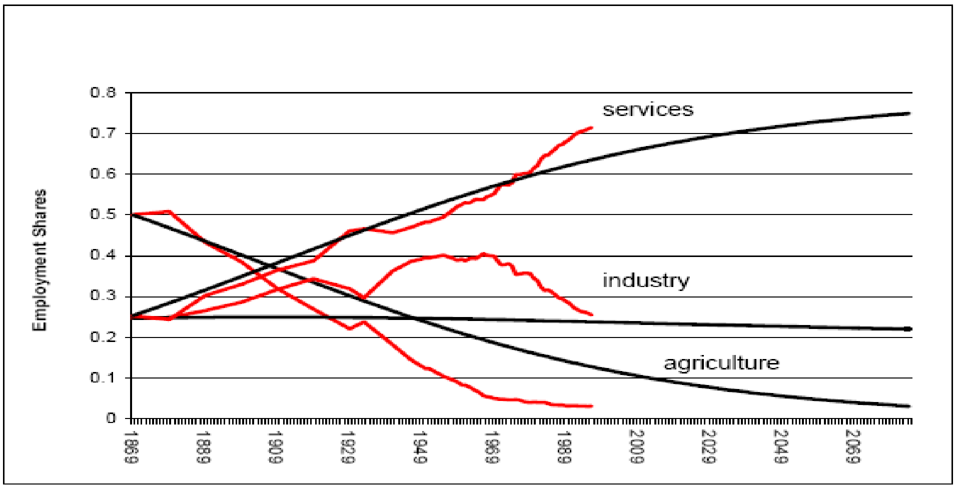


Figure 1: Structural Change in U.S.A. Resource: Ngai and Pissarides (2006)

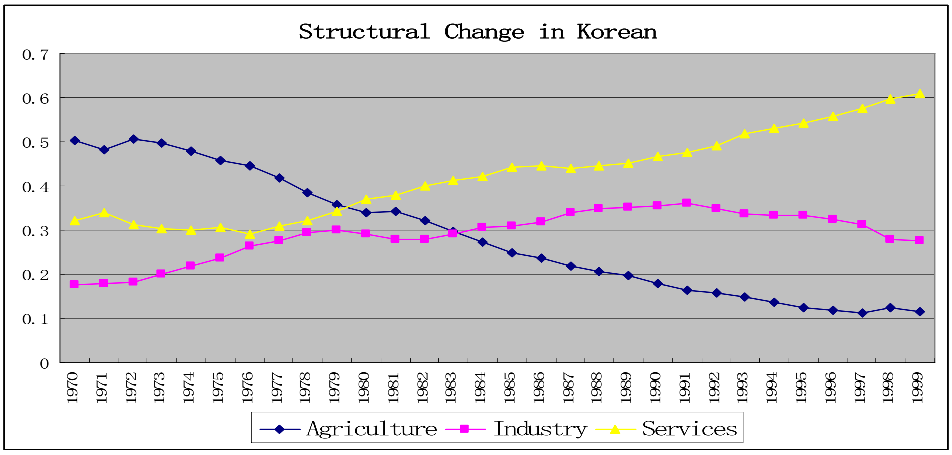


Figure 2: Structural Change in Korean. Resource: OECD industrial data

Although it is valuable to research the consistency of "Kuznets facts" with "Kaldors facts", the essential objective faced by the theory of structural change is to explain the crucial phenomena of non-balanced growth between different industrial sectors and its impact on the whole economy. This insistency on the essential objective is of the essence in the sense that there still exists some very important phenomenon of structural change failed to be explained by the current theory. To illustrate this point, let us look at the empirical facts. Figure 1 reveals the history of structural change for postwar U.S.: a falling employment share of agriculture, a rising share of services and a rising share before the 1960s but decreasing share afterwards in industry. Figure 2 shows the similar trend of structural change in Korean. In addition, the historical OECD evidence presented by Kuznets (1966) and Maddison (1980) also provides almost completely the same picture of structural change as that of U.S. and Korean. In these empirical facts, what impresses us most is not the rising share of services or the falling share of agriculture, but the hump-shaped, that is, initial rising and later falling, share of industry. We emphasize this hump-shaped share of industry in respect that it corresponds to the two impressive stages during the economic development, that is, the stage of industrialization mainly characterized by the massive flow of production factor from agriculture to industry before the peak of the hump-shaped share, and of post-industrialization mainly characterized by the massive flow of production factor into services behind that peak. For the U.S.A, the turning point from industrialization to post-industrialization emerged in the 1960s; as for Korean, it is in the 1990s that Korean began its transition from industrialization to post-industrialization; and as far as our China is concerned, it is well known that we still be in the stage of industrialization.

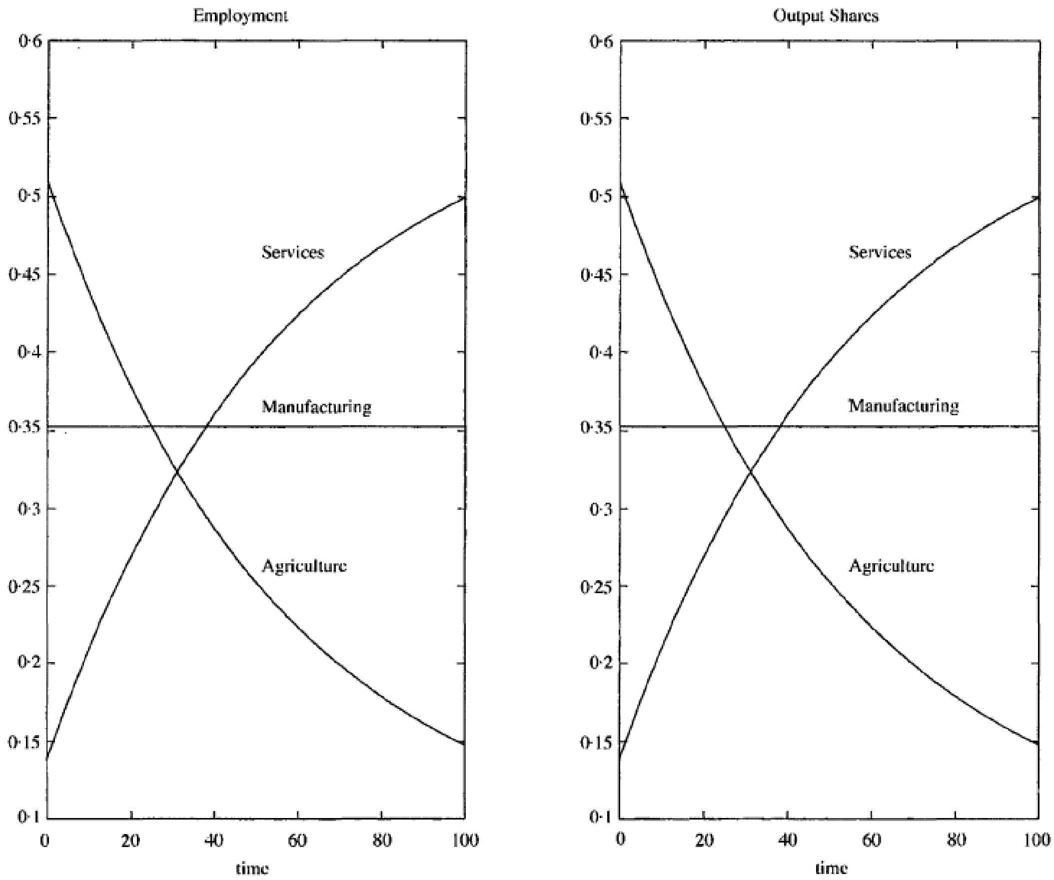


Figure 3, Structural change predicted by Kongsamut, Rebelo and Xie (2001).

Most literature on structural change are of two-sector model and thus simply can but focus on one stage of structural change of industry, either industrialization or post-industrialization. There are mainly two class of literature, one of which emphasizes the role of non-homothetic preferences consistent with Engel’s law of or hierarchy of needs in inducing in structural change and is of the mainstream (Murphy, Shleifer and Vishny (1989), Matsuyama (1992), Echevarria (1997), Laitner (2000),Kongsamut, Rebelo and Xie (2001), Caselli and Coleman (2001), Gollin, Parente and Rogerson (2002),Matsuyama (2005)); and another of which is proposed by Baumol (1967), emphasizing structural change resulting from differential productivity differences between different sectors. The former literature explain industrialization by assuming that consumer has higher elasticity of income for industrial goods relative to agricultural goods, and post-industrialization by assuming consumer has higher elasticity of income for service goods to industrial goods. The later literature assume that the labor-intensive service goods and the capital-intensive manufacturing goods are gross substitutes, and thus the higher growth rate of productivity in capital-intensive sector will result in the flow of production factor from the "progressive " sector, i.e. manufacturing, into the relatively "laggard " sector, i.e., services.

There is some latest literature on structural change model with three or even more sectors, but they merely depict the flow of production factor from agriculture into services, all failed to characterize the hump-shaped employment or capital share of industry in the whole economy. That is, in these models, we can not see the stage of industrialization or of post-industrialization. For example, in the paper by Kongsamut, Rebelo and Xie (2001), their Geary-Stone utility function results in a rising share of services, falling share of agriculture and constant share of industry (see figure 3); in Ngai and Pissarides's paper (2006) that extend Baumol's model (1967) , the labor or capital employment share is possible to be decreasing, constant or increasing, but impossible to be initially rising and then falling (see figure 1).

This paper constructs a three-sector model of structural change with hump-shaped employment share of industry. The main idea is inspired by Baumol's thoughts. Our critical assumptions are the gross substitutity between the industrial goods and the agricultural goods, and the gross complementarity between the service goods and the compound goods of the industrial goods and the agricultural goods. The substitutity of industry and agriculture results in the flow of factor from lower-growing agriculture into higher-growing industry, and the complementarity of the compound industry and services lead to the flow of factor from the higher-growing compound industry into lower-growing services. The net result of these two effects will bring about a hump-shaped employment share of industry.

While the direction of structural change is of Baumol's style in our model, the main economic force for structural change is the differences of the three sectors in their capital intensity proposed first by one of our authors Justin Lin (1997) and modeled first by Acemoglu and Guerriieri (2006). Technical progress leads to the updating of endowment structure, that is, the larger stock of capital per capita, which conduces to the growth of capital intensive sectors and thus cause sectoral change in allocation of factors combined with the effect of the substitutiy and complementarity between different goods.

Our model is highly related to the work of Acemoglu and Guerriieri (2006) in the sense that we also use the differences in capital intensity and the updating of endowment structure as the main economic force for structural change. Also, our model is related to that of Ngai and Pissarides (2006) in the sense that we all use the same mechanism proposed by Baumol (1967). But, our model improves greatly on their work in the following aspects: 1, we extend the two-sector model of Acemoglu and Guerriieri (2006) into a three-sector model, including the consistency of structural change in allocation of factors with the asymptotically constant interest rate and aggregate growth rate; 2, we introduce heterogeneous elasticity of substitute between different sectors, which is a natural and meaningful extension for the model of Ngai and Pissarides

(2006); 3, and the most important, our model can predict a hump-shaped employment share of industry and therefore be able to explain both industrialization and post-industrialization in a unified framework..

The rest of the paper is organized as follows. Section 2 specifies the basic assumptions and set up, giving the first order and optimal conditions for our dynamic system. Section 3 determines the direction of structural change for different sectors and gives the dynamic functions representing structural change. Section 4 proves the existence of balanced growth paths and their local stability. Section 5 undertakes a simple calibration to check the hump-shaped employment share of industry. Section 6 concludes.

II. The Basic Model

In our economy, there are three sectors with Cobb-Douglas technologies

$$Y_i = A_i K_i^{\alpha_i} L_i^{1-\alpha_i} \qquad (i = 1, 2, 3), \tag{1}$$

where we assume that the capital intensity across different sectors is different, and specifically, we assume

$$\alpha_1 > \alpha_2 > \alpha_3 . \tag{A 1}$$

This assumption means that the first sector with subscript 1 represents industry with the highest capital intensity, the second sector with subscript 2 represents agriculture with the less capital intensity, and the third sector with subscript 3 represents services with the least capital intensity or highest labor intensity.

The industrial goods and agriculture goods are produced competitively using constant elasticity of substitution (CES) production function with elasticity of substitution between the two goods equal to $\varepsilon > 1$:

$$Y_M = \left[\gamma Y_1^{\frac{\varepsilon-1}{\varepsilon}} + (1-\gamma) Y_2^{\frac{\varepsilon-1}{\varepsilon}} \right]^{\frac{\varepsilon}{\varepsilon-1}} . \tag{2}$$

Here above-unit elasticity of substitution represents the gross substitutability between the industrial goods and the agricultural goods, and for concreteness, we denote Y_M as the compound goods and the sector producing compound goods as the compound sector.

The final goods is produced by combing the above compound goods with the service goods with an elasticity of substitution $\eta < 1$:

$$Y = \left[\phi Y_M^{\frac{\eta-1}{\eta}} + (1-\phi) Y_3^{\frac{\eta-1}{\eta}} \right]^{\frac{\eta}{\eta-1}} . \tag{3}$$

Substituting equation (3) into (4) immediately implies a two-level CES production function for the final goods which has been used by Krusell, Ohanian, Rios-Rull, and Violante (2000) and Caselli and Coleman (2006).

To eliminate the effect of differences in productivity growth on structural change, we assume the productivity or technology of the three sectors all grow at the same rate, i.e.,

$$A_i(t) = e^{m\cdot t}\cdot A_i(0) \qquad (i = 1,2,3). \qquad (4)$$

In addition, we assume that there is an exponential population growth

$$L(t) = e^{n\cdot t}\cdot L(0). \qquad (5)$$

All factor markets are competitive, and the market clearing for the two factors implies

$$K = K_M + K_3 \qquad (6)$$

and

$$L = L_M + L_3, \qquad (7)$$

where K_M and L_M respectively denote the capital and labor stock used to produce both the industrial and agricultural goods, and therefore $K_M = K_1 + K_2$ and $L_M = L_1 + L_2$.

We assume that all households have constant relative risk aversion (CRRA) preferences over total household consumption (rather than per capita consumption), thus it implies that the economy admits a representative agent with CRRA preference:

$$\int_0^\infty \frac{C^{1-\theta}-1}{1-\theta}\cdot e^{-\rho t}dt \qquad (8)$$

where $C(t)$ is aggregate consumption at time t, ρ is the rate of time preferences and $\theta \geq 0$ is the inverse of the intertemporal elasticity of substitution or the coefficient of relative risk aversion.

The budget constraint for the economy or the social planner is:

$$\dot{K} = Y - C \qquad (9)$$

According to the above setup, the social optimization problem can be characterized by a system of optimal control with objective function (8) subject to the budget constraint (9), the market clear conditions (6) and (7).

The Hamilton function of this system can be written as:

$$H = \frac{C^{1-\theta}-1}{1-\theta}\cdot e^{-\rho t} + \lambda\left\{\left[\phi(\gamma Y_1^{\frac{\varepsilon-1}{\varepsilon}} + (1-\gamma)Y_2^{\frac{\varepsilon-1}{\varepsilon}})^{\frac{\varepsilon}{\varepsilon-1}\frac{\eta-1}{\eta}} + (1-\phi)Y_3^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}} - C\right\}, \qquad (10)$$

$$+\mu(K - K_1 - K_2 - K_3) + \upsilon(L - L_1 - L_2 - L_3)$$

and then the first order and optimal conditions can be written as :

$$\frac{\partial H}{\partial C} = C^{-\theta}e^{-\rho t} - \lambda = 0 \qquad (11)$$

$$\frac{\partial H}{\partial K_1} = \lambda\phi\gamma\alpha_1(\frac{Y}{Y_M})^{\frac{1}{\eta}}(\frac{Y_M}{Y_1})^{\frac{1}{\varepsilon}}\frac{Y_1}{K_1} - \mu = 0 \quad (12)$$

$$\frac{\partial H}{\partial K_2} = \lambda\phi\alpha_2(1-\gamma)(\frac{Y}{Y_M})^{\frac{1}{\eta}}(\frac{Y_M}{Y_2})^{\frac{1}{\varepsilon}}\frac{Y_2}{K_2} - \mu = 0 \quad (13)$$

$$\frac{\partial H}{\partial K_3} = \lambda\alpha_3(1-\phi)(\frac{Y}{Y_3})^{\frac{1}{\eta}}\frac{Y_3}{K_3} - \mu = 0 \quad (14)$$

$$\frac{\partial H}{\partial L_1} = \lambda\phi\gamma(1-\alpha_1)(\frac{Y}{Y_M})^{\frac{1}{\eta}}(\frac{Y_M}{Y_1})^{\frac{1}{\varepsilon}}\frac{Y_1}{L_1} - \nu = 0 \quad (15)$$

$$\frac{\partial H}{\partial L_2} = \lambda\phi(1-\gamma)(1-\alpha_2)(\frac{Y}{Y_M})^{\frac{1}{\eta}}(\frac{Y_M}{Y_2})^{\frac{1}{\varepsilon}}\frac{Y_2}{L_2} - \nu = 0 \quad (16)$$

$$\frac{\partial H}{\partial L_3} = \lambda(1-\phi)(1-\alpha_3)(\frac{Y}{Y_3})^{\frac{1}{\eta}}\frac{Y_3}{L_3} - \nu = 0\,, \quad (17)$$

$$\dot{\lambda} = -\frac{\partial H}{\partial K} = -\mu\,; \quad (18)$$

and the transversality condition is:

$$\lim_{t\to\infty}\lambda(t)K(t) = 0\,. \quad (19)$$

III. Structural Change

Let us introduce the following notations for the fractions of capital and labor in different sector:

$$k_1 = \frac{K_1}{K},\ \ k_3 = \frac{K_3}{K},\ \ k_m = \frac{K_1}{K_M}\,,$$

$$l_1 = \frac{L_1}{L},\ \ l_3 = \frac{L_3}{L},\ \ l_m = \frac{L_1}{L_M}\,.$$

Obviously, we have

$$k_1 = k_m(1-k_3) \quad (20)$$

and

$$l_1 = l_m(1-l_3)\,, \quad (21)$$

and therefore, it is sufficient to characterize the structural change of the whole economy as long as we describe the dynamic behaviors of capital fractions in any two sectors. In this setup, we will focus on k_m, the capital fraction of industry relative to the compound sector, and k_3, the fraction of capital in services. Based on them, we then could depict the dynamics of industry's capital fraction relative to the whole economy in terms of equation (20).

Lemma 1. we can denote labor fractions of industry relative to the compound sector and of services relative to the whole economy as the express of their capital fractions, that is, l_m and l_3 as the expression of k_m and k_3:

$$l_m = \frac{\alpha_2(1-\alpha_1)k_m}{\alpha_2(1-\alpha_1)k_m + \alpha_1(1-\alpha_2)(1-k_m)} \tag{22}$$

$$l_3 = \left[\frac{(1-\alpha_1)\alpha_2\alpha_3 k_m + \alpha_1(1-\alpha_2)\alpha_3(1-k_m)}{\alpha_1\alpha_2(1-\alpha_3)} \cdot \frac{1-k_3}{k_3} + 1\right]^{-1}. \tag{23}$$

Proof. Combing equations (12) and (15), we obtain

$$\frac{u}{v} = \frac{\alpha_1}{1-\alpha_1} \cdot \frac{L_1}{K_1}, \tag{24}$$

and then combing equation (13) and (16) yields

$$\frac{u}{v} = \frac{\alpha_2}{1-\alpha_2} \cdot \frac{L_2}{K_2}. \tag{25}$$

At last, combing the above two equations and using the definitions of l_m and k_m gives

$$\frac{1-l_m}{l_m} = \frac{1-\alpha_2}{1-\alpha_1} \cdot \frac{\alpha_1}{\alpha_2} \cdot \frac{1-k_m}{k_m}, \tag{26}$$

and reforming it gives (22).

To obtain equation (23), firstly combing equations (14) and (17) yields

$$\frac{u}{v} = \frac{\alpha_3}{1-\alpha_3} \cdot \frac{L_3}{K_3}, \tag{27}$$

and then applying the similar process in the obtainment of equation (26) gives

$$\frac{k_1}{l_1} = \frac{1-\alpha_3}{1-\alpha_1} \cdot \frac{\alpha_1}{\alpha_3} \cdot \frac{k_3}{k_1}$$

which can be easily reformed into

$$\frac{k_m}{l_m} \cdot \frac{1-k_3}{k_3} = \frac{\alpha_1(1-\alpha_3)}{\alpha_3(1-\alpha_1)} \cdot \frac{1-l_3}{l_3} \tag{28}$$

by using $k_1 = k_m(1-k_3)$ and $l_1 = l_m(1-l_3)$.

At last, substituting the expression of l_m in (23) into (28) and after some transformation, we get (23). ■

Based on the lemma 1, we can get the following important results.

Lemma 2. In equilibrium, we have the following inequality

$$k_m > l_m \quad \text{and} \quad l_3 > k_3 .$$

Proof. Reforming the equation (26) gives

$$k_m - l_m = \frac{\alpha_1 - \alpha_2}{(1-\alpha_1)\alpha_2}(1-k_m)l_m , \tag{29}$$

which is obviously larger than zero because we have assumed that α_1 is strictly larger than α_2 and that k_m is strictly smaller than one in terms of its definition.

As long as $k_m > l_m$, the following inequality will be naturally established based on the equation (28)

$$\frac{k_3(1-l_3)}{l_3(1-k_3)} < \frac{\alpha_3(1-\alpha_1)}{(1-\alpha_3)\alpha_1} ,$$

which directly imply $k_3 < l_3$ because we have assumed that $\alpha_1 > \alpha_3$. ■

The economic meaning of lemma 2 is that in equilibrium, the labor-intensive sectors should employ more labor than capital fraction, and the capital-intensive sectors should employ more capital than labor fraction. Specifically, services with the lowest capital intensity or the highest labor intensity should employ higher fraction of labor relative to its fraction of capital employment in the whole economy, while in the compound sector, industry has the relatively higher capital intensity than agriculture has and thus it is reasonable for it to employ more fraction of capital to its fraction of labor.

Proposition 1. The dynamic behavior of the capital faction of industry relative to the compound sector can be characterized as the following expression:

$$\frac{\dot{k}_m}{k_m} = \frac{(\varepsilon-1)(\alpha_1-\alpha_2)(1-k_m)}{1+(\varepsilon-1)(\alpha_1-\alpha_2)(1-l_3)(k_m-l_m)} \cdot \left[(\frac{\dot{K}}{K} - n) + \frac{(l_3-k_3)}{1-k_3} \cdot \frac{\dot{k}_3}{k_3} \right]. \tag{31}$$

Proof. Combing equations (2), (12) and (13) gives

$$\frac{Y_1}{Y_2} = \left[\frac{\alpha_2(1-\gamma)}{\alpha_1 \gamma} \cdot \frac{k_m}{1-k_m} \right]^{\frac{\varepsilon}{\varepsilon-1}} . \tag{32}$$

Next taking logs of both sides in equation (23) yields

$$\frac{\dot{Y}_1}{Y_1} - \frac{\dot{Y}_2}{Y_2} = \frac{\varepsilon}{\varepsilon-1} \cdot \frac{\dot{k}_m}{k_m} \cdot \frac{1}{1-k_m} \tag{33}$$

Differentiating equation (1) gives the expressions of growth rate in the sector of industry and agriculture, and then subtracting $\frac{\dot{Y}_2}{Y_2}$ from $\frac{\dot{Y}_1}{Y_1}$, we get

$$\frac{\dot{Y}_1}{Y_1}-\frac{\dot{Y}_2}{Y_2}=(\alpha_1-\alpha_2)\cdot(\frac{\dot{K}_M}{K_M}-\frac{\dot{L}_M}{L_M})+(\alpha_1+\frac{\alpha_2}{1-k_m})\cdot\frac{\dot{k}_m}{k_m}+(1-\alpha_1+\frac{1-\alpha_2}{1-l_m})\cdot\frac{\dot{l}_m}{l_m}. \qquad (34)$$

According to our denotations of K_m and L_m, it is easy to show that

$$\frac{\dot{K}_M}{K_M}-\frac{\dot{L}_M}{L_M}=\frac{\dot{K}}{K}-n-(\frac{\dot{k}_3}{1-k_3}-\frac{\dot{l}_3}{1-l_3}) \qquad (35)$$

Substituting the above equation into (34) gives

$$\begin{aligned}\frac{\dot{Y}_1}{Y_1}-\frac{\dot{Y}_2}{Y_2}&=(\alpha_1-\alpha_2)\cdot(\frac{\dot{K}}{K}-n)-(\alpha_1-\alpha_2)\cdot(\frac{\dot{k}_3}{1-k_3}-\frac{\dot{l}_3}{1-l_3})\\&+(\alpha_1+\frac{\alpha_2}{1-k_m})\cdot\frac{\dot{k}_m}{k_m}+(1-\alpha_1+\frac{1-\alpha_2}{1-l_m})\cdot\frac{\dot{l}_m}{l_m}\end{aligned} \qquad (36)$$

Now we differentiate the both sides of equation (28) and obtain

$$\frac{\dot{l}_3}{1-l_3}=\frac{l_3}{1-k_3}\cdot\frac{\dot{k}_3}{k_3}+\frac{(k_m-l_m)l_3}{1-k_m}\cdot\frac{\dot{k}_m}{k_m} \qquad (37)$$

Substituting (37) into (36) gives

$$\begin{aligned}\frac{\dot{Y}_1}{Y_1}-\frac{\dot{Y}_2}{Y_2}&=(\alpha_1-\alpha_2)\cdot(\frac{\dot{K}}{K}-n)+\frac{(\alpha_1-\alpha_2)(l_3-k_3)}{1-k_3}\cdot\frac{\dot{k}_3}{k_3}\\&+\frac{1-(\alpha_1-\alpha_2)(1-l_3)(k_m-l_m)}{1-k_m}\cdot\frac{\dot{k}_m}{k_m}\end{aligned}. \qquad (38)$$

We can eventually obtain the targeting equation (31) by combing (33) and (38) and making some transformation. ■

Proposition 1 is crucial in that it shows that, given the growth rate of capital fraction in services, industry's capital fraction in the compound sector is monotonically increasing as long as $\alpha_1>\alpha_2$, representing higher capital intensity of industry relative to that of agriculture, and $\varepsilon>1$, representing the gross substitutability between the industrial and agricultural goods. The condition $\alpha_1>\alpha_2$ leads to the increase in the relative output of industry and thus brings about non-balanced growth between industry and agriculture. Furthermore, the condition $\varepsilon>1$ enhances the tendency of substitution of industrial goods for agricultural goods and thus induces the flow of factors from agriculture into industry. So, the comparison of α_1 and α_2 represents and controls the economic force for structural change, and the elasticity of substitution ε determines the direction of structural change.

The dynamic behavior of capital fraction in services, k_3, is relatively more complicated. We will describe it with the following proposition and corollary.

Proposition 2. The dynamic behavior of the capital faction of the service sector relative to the whole economy can be characterized as the following expression:

$$\begin{aligned}\frac{\dot{k}_3}{k_3}=&\frac{(1-\eta)(1-k_3)}{1-(1-\eta)(\alpha_1-\alpha_3)(l_3-k_3)}\cdot(\alpha_1-\alpha_3)(\frac{\dot{K}}{K}-n)\\&-\frac{(1-\eta)(1-k_3)}{1-(1-\eta)(\alpha_1-\alpha_3)(l_3-k_3)}\frac{\alpha_1(1-\eta)+(\alpha_1-\alpha_2)(\varepsilon-1)k_m}{(1-\eta)(\varepsilon-1)\left[\alpha_1(1-k_m)+\alpha_2k_m\right]}\cdot\frac{\dot{k}_m}{k_m}\\&-\frac{(1-\eta)(1-k_3)}{1-(1-\eta)(\alpha_1-\alpha_3)(l_3-k_3)}\frac{(\alpha_1-\alpha_3)(1-l_3)(k_m-l_m)}{1-k_m}\cdot\frac{\dot{k}_m}{k_m}\end{aligned}. \tag{39}$$

Proof. Combing equations (12) and (14), we have

$$\frac{\phi\gamma\alpha_1}{\alpha_3(1-\phi)}\left(\frac{Y_M}{Y_3}\right)^{1-\frac{1}{\eta}}\left(\frac{Y_M}{Y_1}\right)^{\frac{1}{\varepsilon}-1}=\frac{K_1}{K_3}. \tag{40}$$

Then based on equations (2) and (32), it is easy to get

$$Y_M=\left(\gamma+\frac{\gamma\alpha_1}{\alpha_2}\frac{1-k_m}{k_m}\right)^{\frac{\varepsilon}{\varepsilon-1}}Y_1. \tag{41}$$

Substituting equation (41) into (40) and making some simple transformation yields

$$\frac{\phi\alpha_1}{\alpha_3(1-\phi)}\gamma^{\frac{\varepsilon(1-\eta)}{(1-\varepsilon)\eta}}\left(1+\frac{\alpha_1}{\alpha_2}\frac{1-k_m}{k_m}\right)^{\frac{\varepsilon-\eta}{(1-\varepsilon)\eta}}\left(\frac{Y_1}{Y_3}\right)^{1-\frac{1}{\eta}}=\frac{k_m(1-k_3)}{k_3}. \tag{42}$$

Differentiating the above equation gives

$$-\frac{\varepsilon-\eta}{(1-\varepsilon)\eta}\cdot\frac{\alpha_1}{\alpha_1(1-k_m)+\alpha_2k_m}\cdot\frac{\dot{k}_m}{k_m}+(1-\frac{1}{\eta})(\frac{\dot{Y}_1}{Y_1}-\frac{\dot{Y}_3}{Y_3})=\frac{\dot{k}_m}{k_m}-\frac{\dot{k}_3}{k_3}\cdot\frac{1}{1-k_3}. \tag{43}$$

Using the similar process of obtainment of the expression of $\frac{\dot{Y}_1}{Y_1}-\frac{\dot{Y}_2}{Y_2}$, we can get

$$\begin{aligned}\frac{\dot{Y}_1}{Y_1}-\frac{\dot{Y}_3}{Y_3}=&(\alpha_1-\alpha_3)\cdot(\frac{\dot{K}}{K}-n)+\frac{1-k_m-(\alpha_1-\alpha_3)(1-l_3)(k_m-l_m)}{1-k_m}\cdot\frac{\dot{k}_m}{k_m}\\&-\frac{1+(\alpha_1-\alpha_3)(k_3-l_3)}{1-k_3}\cdot\frac{\dot{k}_3}{k_3}\end{aligned}. \tag{44}$$

Equation (39) can be obtained by combining equation (43) and (44). ■

Similar to that of k_m, the dynamic behavior of k_3 is primarily determined by the parameters representing capital intensity and the elasticity of substitution. But here, services is labor-intensive relative to agriculture and industry, and thus the updating of endowment structure or capital deepening is unfavorable to the increase of output in services. The under-unit elasticity of substitution implies the gross complementarity between the compound goods and the service goods, which tends to increase the price of goods in the slow-growing sector and then simultaneously educe the flow of factors from the compound sector to services.

Although the above analysis on proposition 1 and proposition 2 indicate that, given the growth of k_3 and k_m, the updating of endowment structure is inclined to bring about the rise of k_m and k_3, equation (39) also shows that the growth of k_3 is negatively related to that of k_m. A higher growth rate of k_m tends to decrease the growth rate of k_3. So, the net effect is still uncertain before we can determine the sign and scale of k_m's growth function. In the following corollary, we show that both k_3 and k_m increase with the updating of endowment structure under some reasonable and loosing conditions.

For simplicity, we denote:

$$B_1 = \frac{(1-\eta)(1-k_3)}{1-(1-\eta)(\alpha_1-\alpha_3)(l_3-k_3)},$$

$$B_2 = \frac{(\varepsilon-1)(1-k_m)}{1+(\varepsilon-1)(\alpha_1-\alpha_2)(1-l_3)(k_m-l_m)},$$

$$B_3 = \frac{\alpha_1(1-\eta)+(\alpha_1-\alpha_2)(\varepsilon-1)k_m}{(1-\eta)(\varepsilon-1)\left[\alpha_1(1-k_m)+\alpha_2 k_m\right]} + \frac{(\alpha_1-\alpha_3)(1-l_3)(k_m-l_m)}{1-k_m},$$

and

$$B_4 = \frac{(\alpha_1-\alpha_2)(l_3-k_3)}{1-k_3}.$$

Obviously, all of the above equations is positive in terms of our assumptions on ε and η, and comparison of k_m and l_m or l_3 and k_3 in lemma 2. In terms of these denotations, we can rewrite the expression of $\frac{\dot{k}_3}{k_3}$ and $\frac{\dot{k}_m}{k_m}$ as the following compact forms:

$$\frac{\dot{k}_3}{k_3} = B_1(\alpha_1-\alpha_3)(\frac{\dot{K}}{K}-n) - B_1 B_3 \cdot \frac{\dot{k}_m}{k_m}$$

$$\frac{\dot{k}_m}{k_m} = B_2(\alpha_1-\alpha_2)(\frac{\dot{K}}{K}-n) + B_2 B_4 \cdot \frac{\dot{k}_3}{k_3}$$

Reforming the above differentiating functions of $\frac{\dot{k}_3}{k_3}$ and $\frac{\dot{k}_m}{k_m}$ gives

$$\frac{\dot{k}_3}{k_3} = \frac{B_1\left[\alpha_1-\alpha_3-B_2B_3(\alpha_1-\alpha_2)\right]}{1+B_1B_2B_3B_4}\cdot(\frac{\dot{K}}{K}-n), \qquad (45)$$

and

$$\frac{\dot{k}_m}{k_m} = \frac{B_2\left[\alpha_1 - \alpha_2 + B_1 B_4(\alpha_1 - \alpha_3)\right]}{1 + B_1 B_2 B_3 B_4} \cdot \left(\frac{\dot{K}}{K} - n\right). \tag{46}$$

Based on the above forms, we can determine the dynamic behavior of k_m and k_3.

Corollary 1. As long as $\frac{1-\eta}{\varepsilon - 1} > \frac{\alpha_1 - \alpha_2}{\alpha_2}$, the fraction of capital in the service sector relative to the whole economy is monotonically increasing until it reach to its upper limit, that is, we have:

$$\frac{\dot{k}_3}{k_3} > 0;$$

and if assumption A1 holds, then we have

$$\frac{\dot{k}_m}{k_m} > 0.$$

Proof. $\frac{\dot{k}_m}{k_m} > 0$ directly results from equation (46) as long as we notice that $\alpha_1 > \alpha_2 > \alpha_3$ and $B_i > 0$ for all i.

To establish the growth direction of k_3, we first show

$$\frac{\dot{k}_3}{k_3} > 0 \Leftrightarrow B_2 B_3 < \frac{\alpha_1 - \alpha_3}{\alpha_1 - \alpha_2}$$

according to equation (45).

Multiplying B_2 by B_3 yields

$$B_2 B_3 = \frac{\dfrac{\alpha_1(1-\eta)(1-k_m) + (\alpha_1 - \alpha_2)(\varepsilon - 1)k_m(1-k_m)}{(1-\eta)\left[\alpha_1(1-k_m) + \alpha_2 k_m\right]} + (\varepsilon - 1)(\alpha_1 - \alpha_3)(1-l_3)(k_m - l_m)}{1 + (\varepsilon - 1)(\alpha_1 - \alpha_2)(1-l_3)(k_m - l_m)}, \tag{47}$$

which is smaller than

$$\frac{1 + (\varepsilon - 1)(\alpha_1 - \alpha_3)(1-l_3)(k_m - l_m)}{1 + (\varepsilon - 1)(\alpha_1 - \alpha_2)(1-l_3)(k_m - l_m)} \tag{48}$$

as long as

$$\frac{(\alpha_1 - \alpha_2)(\varepsilon - 1)k_m(1-k_m)}{\alpha_2(1-\eta)k_m} < 1. \tag{49}$$

If we assume that $\frac{1-\eta}{\varepsilon - 1} > \frac{\alpha_1 - \alpha_2}{\alpha_2}$, then the inequality (49) holds, and therefore the following inequalities hold

$$B_2B_3 < \frac{1+(\varepsilon-1)(\alpha_1-\alpha_3)(1-l_3)(k_m-l_m)}{1+(\varepsilon-1)(\alpha_1-\alpha_2)(1-l_3)(k_m-l_m)} < \frac{\alpha_1-\alpha_3}{\alpha_1-\alpha_2},$$

and thus our corollary is established. ■

This corollary says that in order to educe the flow of factor into services, we must have a smaller η, that is, a stronger gross complementarity between the compound goods and the service goods.

Next, let us focus on what we are most interested in, the dynamic behavior of industry's capital fraction in the whole economy.

Proposition 3. Under the conditions $k_3(0) < \frac{\alpha_2}{\alpha_2-\alpha_3}$, there exists a unique pair ($\tilde{k}_3$, $\tilde{k}_m$) such that $\frac{\dot{k}_1}{k_1} > 0$ when $k_3 < \tilde{k}_3$ (or $k_m < \tilde{k}_m$) and $\frac{\dot{k}_1}{k_1} < 0$ when $k_3 > \tilde{k}_3$ (or $k_m > \tilde{k}_m$); and furthermore, if condition $\frac{1-\eta}{\varepsilon-1} > \frac{\alpha_1-\alpha_2}{\alpha_3}$ holds, the function of $\frac{\dot{k}_1}{k_1}$ is concave with a positive initial value.

Proof. The full proof should take three steps:

Step1, the sign of $\frac{\dot{k}_1}{k_1}$ corresponds to the sign of $(\varepsilon-1)(\alpha_1-\alpha_2)f(k_m)-k_3g(k_m)$.

Based on the relationship of k_1, k_m and k_3 in equation (20), we have

$$\frac{\dot{k}_1}{k_1} = \frac{\dot{k}_m}{k_m} - \frac{\dot{k}_3}{1-k_3},$$

and then substituting (45) and (46) into it, we get

$$\frac{\dot{k}_1}{k_1} = \frac{B_2(1-k_3)\left[\alpha_1-\alpha_2+B_1B_4(\alpha_1-\alpha_3)\right]-B_1k_3\left[\alpha_1-\alpha_3-B_2B_3(\alpha_1-\alpha_2)\right]}{(1-k_3)(1+B_1B_2B_3B_4)}\cdot\left(\frac{\dot{K}}{K}-n\right). \quad (50)$$

In terms of our denotations for B_1, B_2, B_3 and B_4, some transformation will induce the following equations:

$$\alpha_1-\alpha_2+B_1B_4(\alpha_1-\alpha_3) = \frac{(\alpha_1-\alpha_2)}{1-(1-\eta)(\alpha_1-\alpha_3)(l_3-k_3)}$$

and

$$\alpha_1-\alpha_3-B_2B_3(\alpha_1-\alpha_2) = \frac{\alpha_1(1-\eta)(\alpha_2-\alpha_3)+(\alpha_1-\alpha_2)k_m\left[\alpha_3(1-\eta)-(\varepsilon-1)(\alpha_1-\alpha_2)(1-k_m)\right]}{(1-\eta)\left[\alpha_1(1-k_m)+\alpha_2k_m\right]\left[1+(\varepsilon-1)(\alpha_1-\alpha_2)(1-l_3)(k_m-l_m)\right]}.$$

Then substituting them into equation (50) enables us to rewrite $\frac{\dot{k}_1}{k_1}$ as

$$\frac{(\varepsilon-1)(\alpha_1-\alpha_2)f(k_m)-k_3g(k_m)}{(1+B_1B_2B_3B_4)B_5}\cdot(\frac{\dot{K}}{K}-n)\,, \qquad (51)$$

where

$$B_5=\left[\alpha_1(1-k_m)+\alpha_2k_m\right]\left[1-(1-\eta)(\alpha_1-\alpha_3)(l_3-k_3)\right]\left[1+(\varepsilon-1)(\alpha_1-\alpha_2)(1-l_3)(k_m-l_m)\right]>0,$$

$$f(k_m)=\alpha_1(1-k_m)^2+\alpha_2k_m(1-k_m),$$

and

$$g(k_m)=\alpha_1(1-\eta)(\alpha_2-\alpha_3)+(\alpha_1-\alpha_2)k_m\left[\alpha_3(1-\eta)-(\varepsilon-1)(\alpha_1-\alpha_2)(1-k_m)\right].$$

Therefore, the question whether $\frac{\dot{k}_1}{k_1}$ is larger or smaller than zero transforms the following question whether $(\varepsilon-1)(\alpha_1-\alpha_2)f(k_m)-k_3g(k_m)$ is larger or smaller than zero. Obviously, the properties of $f(k_m)$ and $g(k_m)$ are crucial to analyze the sign of $\frac{\dot{k}_1}{k_1}$.

Step 2: to discuss the properties of $f(k_m)$ and $g(k_m)$.

Differentiating $f(k_m)$, we find that

$$f^{'}(k_m)=-2\alpha_1+\alpha_2+2(\alpha_1-\alpha_2)k_m<0,$$

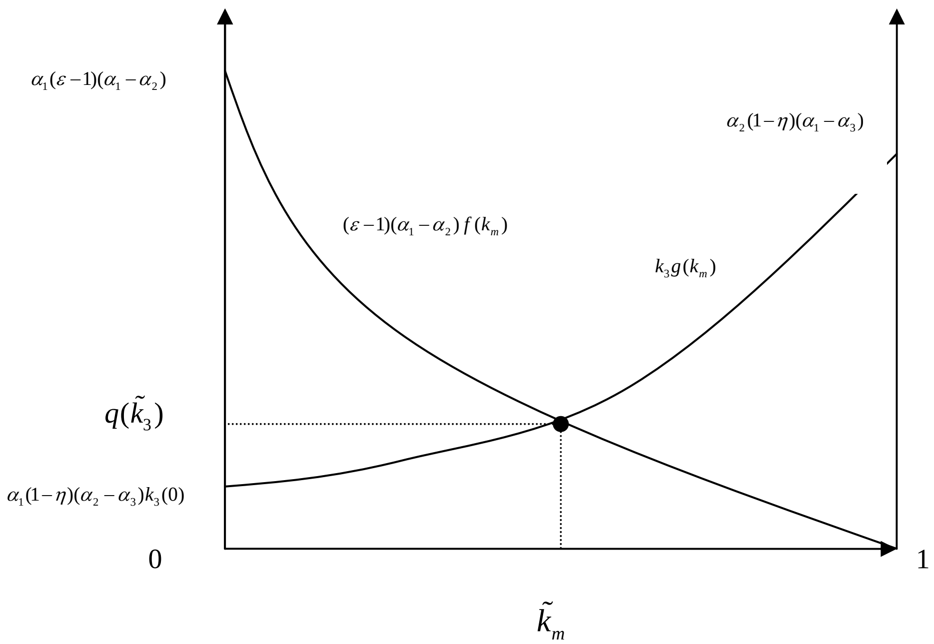
and because $0<k_m<1$, $f^{'}(k_m)<0$ implies that

$$\max f(k_m)=\lim_{k_m\to 0}f(k_m)=\alpha_1, \qquad (52)$$

and

$$\min f(k_m)=\lim_{k_m\to 1}f(k_m)=0. \qquad (53)$$

Thus, the curve of $(\varepsilon-1)(\alpha_1-\alpha_2)f(k_m)$ is monotonically decreasing in the following graph.



Then differentiating $g(k_m)$ twice yields

$$g^{''}(k_m) = 2(\varepsilon-1)(\alpha_1-\alpha_2)^2 > 0,$$

which means $g(k_m)$ is a convex function. It is easy to know that

$$\lim_{k_m \to 1} g(k_m) = \alpha_2(1-\eta)(\alpha_1-\alpha_3), \qquad (54)$$

and

$$\lim_{k_m \to 0} g(k_m) = \alpha_1(1-\eta)(\alpha_2-\alpha_3). \qquad (55)$$

Step 3, the existence and uniqueness of the pair ($\tilde{k}_3$, $\tilde{k}_m$) if there is $k_3(0) < \dfrac{\alpha_2}{\alpha_2-\alpha_3}$.

According to equation (52) to (55), we have

$$\lim_{k_m \to 1} (\varepsilon-1)(\alpha_1-\alpha_2) f(k_m) - k_3 g(k_m) = -\alpha_2(1-\eta)(\alpha_1-\alpha_3) < 0, \qquad (56)$$

and .

$$\lim_{k_m \to 0} (\varepsilon-1)(\alpha_1-\alpha_2) f(k_m) - k_3 g(k_m) = \alpha_1(\varepsilon-1)\left[\alpha_1-\alpha_2-\frac{1-\eta}{\varepsilon-1}(\alpha_2-\alpha_3)k_3\right] > 0 \quad ,$$

(57)

as long as $k_3(0) < \dfrac{\alpha_2}{\alpha_2-\alpha_3}$.

Then, given k_3, added with the convexity of $g(k_m)$ and the decreasing trend of $f(k_m)$, there must exists a unique k_m we denote as $\tilde{k}_3$ satisfying $0 < k_m < 1$ such that $\frac{\dot{k}_1}{k_1} = 0$, $\frac{\dot{k}_1}{k_1} > 0$ when $k_3 < \tilde{k}_3$ and $\frac{\dot{k}_1}{k_1} < 0$ when $k_3 > \tilde{k}_3$. Furthermore, considered that k_3 is monotonically increasing, given $k_3(0)$, there must exists a unique $\tilde{k}_m$ corresponding to $\tilde{k}_3$.

Step 4, the concavity of $\frac{\dot{k}_1}{k_1}$.

Differentiating $g(k_m)$ gives

$$g^{'}(k_m) = (\alpha_1 - \alpha_2)\left[\alpha_3(1-\eta) - (\varepsilon - 1)(\alpha_1 - \alpha_2) + 2(\varepsilon - 1)(\alpha_1 - \alpha_2)k_m\right],$$

which is larger than zero as long as $\frac{1-\eta}{\varepsilon - 1} > \frac{\alpha_1 - \alpha_2}{\alpha_3}$.

If $g^{'}(k_m) > 0$ and thus the function of $g(k_m)$ is increasing monotonically, the expression of $(\varepsilon - 1)(\alpha_1 - \alpha_2) f(k_m) - k_3 g(k_m)$ will be decreasing monotonically in the domain of (0,1), which establish the concavity of $\frac{\dot{k}_1}{k_1}$. ■

IV. The Existence and Stability of the Balanced Growth Paths

The equilibrium behavior of the economy is characterized by balanced growth paths. Usually, the balanced growth path (BGP) is defined as an equilibrium path where the growth rate of consumption exists and is constant, i.e.,

$$\frac{\dot{C}}{C} = g_c^* .$$

From the Euler equation (16) and the first order condition (11), this implies

$$\frac{1}{\theta}\left[\alpha_3(1-\phi)(\frac{Y}{Y_3})^{\frac{1}{\eta}}\frac{Y_3}{K_3} - \rho\right] = g_c^* . \tag{47}$$

Based on the above equation, we can get the following proposition.

Proposition 4. On the balanced growth path, Y , K , C , Y_1 and K_1 grow at the same rate, which is represented by

$$g_Y^* = g_K^* = g_c^* = g_{Y_3}^* = g_{K_3}^* = n + \frac{m}{1-\alpha_3} .$$

Proof. According to equation (1), it is easy to show

$$\frac{Y}{Y_3}=\left[\phi\left(\frac{Y_M}{Y_3}\right)^{\frac{\eta-1}{\eta}}+1-\phi\right]^{\frac{\eta}{\eta-1}},$$

and then substituting the expression of Y_M in (41) into the above equation, we get

$$\frac{Y}{Y_3}=\left[\phi\left(\gamma+\frac{\gamma\alpha_1}{\alpha_2}\frac{1-k_m}{k_m}\right)^{\frac{\varepsilon(\eta-1)}{(\varepsilon-1)\eta}}\left(\frac{Y_1}{Y_3}\right)^{\frac{\eta-1}{\eta}}+1-\phi\right]^{\frac{\eta}{\eta-1}}. \tag{48}$$

Corollary 1 has showed that k_3 converges to 1 gradually, therefore $\frac{Y_1}{Y_3}$ will asymptotically converge to a constant, which means Y will grow asymptotically at the same rate as that of Y_3.

Furthermore, $\frac{Y_1}{K_1}$ also should asymptotically converge to a constant according to equation (47) if the balanced growth path exists. This implies that Y_1, K_1 and thus K (because $\lim_{t\to\infty}k(t)=1$) must grow at the same rate..

It is easy to show that

$$\frac{Y_3}{K_3}=k_3^{\alpha_1-1}l_3^{1-\alpha_1}A_3K^{\alpha_3-1}L^{1-\alpha_3}, \tag{49}$$

which means that on the balanced growth path, capital stock K, must grow at the same rate as that of $A_3^{\frac{1}{1-\alpha_3}}L$, i.e.,

$$g_K^*=\frac{\dot{K}}{K}=n+\frac{m}{1-\alpha_3}.$$

■

Proposition 4 implies that the equilibrium behavior of this economy can be represented by a system of autonomous non-linear differential equations in three variables:

$$c=\frac{C}{A_3^{\frac{1}{1-\alpha_3}}L},\quad x=\frac{K}{A_3^{\frac{1}{1-\alpha_3}}L},\quad k_3 \text{ and } k_m.$$

Here c is the level of consumption normalized by population and technology f the capital-intensive sector and is the only control variable; x is the capital stock normalized by the same denominator; k_3 and k_m determine the allocation of capital between the three sectors.

These three are state variables with given initial conditions $x(0)$, $k_3(0)$ and $k_m(0)$.

The dynamic equilibrium conditions then translate into the following equations:

$$
\begin{aligned}
\frac{\dot{c}}{c} &= \frac{1}{\theta}\left[\alpha_3(1-\phi)\left(\frac{Y}{Y_3}\right)^{\frac{1}{\eta}} l_3^{1-\alpha_3}\ (k_3 x)^{-(1-\alpha_3)} - \rho\right] - \frac{m}{1-\alpha_3} - n \\
\frac{\dot{x}}{x} &= l_3^{1-\alpha_3} k_3^{\alpha_3} x^{-(1-\alpha_3)} \cdot \frac{Y}{Y_3} - \frac{c}{x} - \frac{m}{1-\alpha_3} - n \\
\frac{\dot{k}_3}{k_3} &= \frac{(1-\eta)(1-k_3)}{1-(1-\eta)(\alpha_1-\alpha_3)(l_3-k_3)} \cdot (\alpha_1-\alpha_3)\left(\frac{\dot{x}}{x} + \frac{m}{1-\alpha_3}\right) \\
&\quad - \frac{(1-\eta)(1-k_3)}{1-(1-\eta)(\alpha_1-\alpha_3)(l_3-k_3)} \frac{\alpha_1(1-\eta)+(\alpha_1-\alpha_2)(\varepsilon-1)k_m}{(1-\eta)(\varepsilon-1)\left[\alpha_1(1-k_m)+\alpha_2 k_m\right]} \cdot \frac{\dot{k}_m}{k_m} \\
&\quad - \frac{(1-\eta)(1-k_3)}{1-(1-\eta)(\alpha_1-\alpha_3)(l_3-k_3)} \frac{(\alpha_1-\alpha_3)(1-l_3)(k_m-l_m)}{1-k_m} \cdot \frac{\dot{k}_m}{k_m} \\
\frac{\dot{k}_m}{k_m} &= \frac{(\varepsilon-1)(\alpha_1-\alpha_2)(1-k_m)}{1+(\varepsilon-1)(\alpha_1-\alpha_2)(1-l_3)(k_m-l_m)} \cdot \left[\left(\frac{\dot{x}}{x} + \frac{m}{1-\alpha_3}\right) + \frac{(l_3-k_3)}{1-k_3} \cdot \frac{\dot{k}_3}{k_3}\right]
\end{aligned}
\tag{50}
$$

where $\frac{Y}{Y_3}$ is given by equation (48) and l_3 by equation (23).

Obviously, the definitions of c, x, k_3 and k_m have ensured that the steady-state equilibrium in the above dynamic system corresponds to the balanced growth path in our economy, i.e., c, x, k_3 and k_m must be constant in the BGP equilibrium. According to equation (48), $\frac{Y}{Y_3}$ will converge to $(1-\phi)^{\frac{\eta}{\eta-1}}$, and in terms of our proposition 1 and corollary 1, k_3, k_m and thus l_3 and l_m will converge to 1. Therefore, these steady-state values of dynamic system (50) are given by

$$k_3^* = k_m^* = 1,$$

$$x^* = \left[\frac{\theta\left(n+\frac{m}{1-\alpha_3}\right)+\rho}{\alpha_3(1-\phi)^{\frac{\varepsilon}{\eta-1}}}\right]^{-\frac{1}{1-\alpha_3}},$$

and

$$c^* = (1-\phi)^{\frac{\eta}{\eta-1}}(x^*)^{\alpha_3} - \left(n+\frac{m}{1-\alpha_3}\right)x^*.$$

To complete our description of this economy with two different capital-intensive sectors, now we should show that c, x, k_3 and k_m will converge to c^*, x^*, k_3^* and k_m^* in a saddle path, i.e., the system (50) is locally stable. The next proposition states that this is the case.

Proposition 5. The non-linear system (31) is locally stable, in the sense that in the neighborhood of c^*, x^*, k_3^* and k_m^*, there is a unique two-dimensional manifold of solutions that converge to c^*, x^*, k_3^* and k_m^*.

Proof. Rewrite the system (50) in a more compact form as

$$\dot{y} = f(y),$$

where $y = (c, x, k_3, k_m)'$. To investigate the dynamics of the system (31) in the neighborhood of the steady state, consider the linear system

$$z = f(y^*)z,$$

where $z = y - y^*$ and y^* such that $f(y^*) = 0$, where $J(y^*)$ is Jacobian matrix of $f(y)$ evaluated at y^*.

Before we proceed, let us give important results which will be useful in the following algebra operation,

$$\frac{\partial l_3}{\partial k_m}|_{y=y*} = 0,$$

$$\frac{\partial l_3}{\partial k_3}|_{y=y*} = \frac{(1-\alpha_1)\alpha_3}{\alpha_1(1-\alpha_3)},$$

and

$$\frac{\partial l_m}{\partial k_m}|_{y=y*} = \frac{\alpha_1(1-\alpha_2)}{\alpha_2(1-\alpha_1)}.$$

Differentiation and some algebra enable us to write this Jacobian matrix as

$$J(y^*) = \begin{pmatrix} a_{cc} & a_{cx} & a_{ck_3} & a_{ck_m} \\ a_{xc} & a_{xx} & a_{xk} & a_{xk_m} \\ a_{k_3c} & a_{k_3x} & a_{k_3k_3} & a_{k_3k_m} \\ a_{k_mc} & a_{k_mx} & a_{k_mk_3} & a_{k_mk_m} \end{pmatrix}$$

where

$$a_{cc} = a_{k_3c} = a_{k_mc} = a_{k_3x} = a_{k_mx} = a_{ck_m} = a_{k_3k_m} = a_{k_mk_3} = 0$$

$$a_{cx} = -\frac{\alpha_3(1-\alpha_3)}{\theta}(1-\phi)^{\frac{\eta}{\eta-1}} c^*(x^*)^{-1-(1-\alpha_3)}$$

$$a_{ck_3} = -\frac{\alpha_3(\alpha_1-\alpha_3)}{\theta\alpha_1}(1-\phi)^{\frac{\eta}{\eta-1}} c^*(x^*)^{-(1-\alpha_1)}$$

$$a_{xc} = -1$$

$$a_{xx} = \alpha_3(1-\phi)^{\frac{\eta}{\eta-1}}(x^*)^{\alpha_3-1} - (n+\frac{m}{1-\alpha_3})$$

$$a_{xk_3} = \frac{\alpha_3(x^*)^{\alpha_3}(1-\phi)^{\frac{\eta}{\eta-1}}}{\alpha_1}$$

$$a_{k_3k_3} = -\frac{m(1-\eta)(\alpha_1-\alpha_3)}{1-\alpha_3}$$

$$a_{k_mk_m} = -\frac{m(\varepsilon-1)(\alpha_1-\alpha_2)}{1-\alpha_3}$$

The determinant of the Jacobian matrix is det($J(y^*)$)= $a_{k_mk_m}A_{k_mk_m}$, where $A_{k_mk_m}$ is the algebraic cofactor of $a_{k_mk_m}$. It is easy to show that $A_{k_mk_m} = -a_{cx}a_{xc}a_{k_3k_3}$ and therefore det($J(y^*)$)= $-a_{cx}a_{xc}a_{k_3k_3}a_{k_mk_m}$. The fact that a_{cx} , a_{xc} , $a_{k_3k_3}$ and $a_{k_mk_m}$ are all negative immediately means that det($J(y^*)$)<0.

To establish the local stability of this system, we should show that three of them are negative and one is positive. Let ψ denotes the vector of the eigenvalues of this matrix, then the characteristic equation is given by

$$\det(J(y^*) - \psi I) = 0 .$$

Using cofactor of expansion, we can expand the characteristic equation as:

$$(a_{k_mk_m} - \psi)\tilde{A}_{k_mk_m} = 0 ,$$

where $\tilde{A}_{k_mk_m} = (a_{k_3k_3} - \psi)\left[\psi(a_{xx} - \psi) + a_{cx}a_{xc}\right]$. Thus, we have

$$(a_{k_mk_m} - \psi)(a_{k_3k_3} - \psi)\left[\psi(a_{xx} - \psi) + a_{cx}a_{xc}\right] = 0 .$$

This expression shows that one of the eigenvalue is equal to $a_{k_3k_3}$ and another is equal to $a_{k_mk_m}$, which are both negative, so there must be three negative and one positive eigenvalues as a result that the determinant of the Jacobian matrix $\det(J(y^*))$ is negative. This establishes the existence of a unique two-dimensional manifold of solutions in the neighborhood of this steady statet and thus local (saddle-path) stability holds ■

This proposition implies that the balanced growth path equilibrium is locally stable, i.e., when the initial values of capital, labor and technology are not too far away from the balanced growth path, the economy will indeed converge to this equilibrium, with non-balanced growth at the sectoral level and constant interest rate at the aggregate.

V. A Simple Calibration

VI. Conclusion