Economic Theory **21**, 263–292 (2003)

Economic Theory

Speculative trading with rational beliefs and endogenous uncertainty*

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Received: March 15, 2001; revised version: April 26, 2002

Summary. This paper introduces the framework of rational beliefs of Kurz (1994), which makes the assumptions of heterogeneous beliefs of Harrison and Kreps (1978) and Morris (1996) more plausible. Agents hold diverse beliefs that are "rational" in the sense of being compatible with ample observed data. In a non-stationary environment the agents only learn about the stationary measure of observed data, but their beliefs can remain non-stationary and diverse. Speculative trading then stems from disagreements among traders. In a Markovian framework of dividends and beliefs, we obtain analytical results to show how the speculative premium depends on the extent of heterogeneity of beliefs. In addition, we demonstrate that there exists a unique Rational Belief Equilibrium (RBE) generically with endogenous uncertainty (as defined by Kurz and Wu, 1996) and that the RBE price is higher than the rational expectation equilibrium price (REE) under some general conditions.

Keywords and Phrases: Speculation, Asset pricing, Rational beliefs, Endogenous uncertainty.

JEL Classification Numbers: D84, G12.

* We are deeply grateful to Mordecai Kurz for his constant encouragement and inspiring guidance over the years. We wish to express our gratitude to an anonymous referee for the very valuable comments provided. We also thank Kenneth Arrow, Peter Hammond, Roko Aliprantis and Nicholas Yannelis for their helpful suggestions and Academia Sinica and the National Science Council of the R.O.C. for their indispensable support.

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1 Introduction

Speculation has been a major research topic for economists, especially in light of recent financial crises and speculative attacks on currency and stock markets. Faced with a similarly turbulent world sixty years ago, Lord Keynes brought to public's attention the relationship between speculation and subjective expectations, comparing the determination of stock prices to a "beauty contest." Investors are "concerned, not with what an investment is really worth to a man who buys it for keeps, but with what the market will value it at, under the mass psychology, three months or a year hence."(Keynes, 1936). According to Kaldor (1939), speculation may be defined as "the purchase (or sale) of goods with a view to resale (or repurchase) at a later date ...." Such speculative behavior cannot exist in a world of complete markets or rational expectations (see Arrow, 1953; Feiger, 1976; Tirole, 1982), where investors do not change their asset holding even when markets reopen later. So the appropriate framework for the study of speculative trading is that of incomplete financial markets with sequential trading; speculative trading can then stem from disagreements among investors. The purpose of this paper is to probe further into the relationship between speculation and subjective valuation and to provide a rigorous foundation for a theory on asset pricing with speculative trading.

There are at least three approaches to modeling disagreements and speculation.[1] The first is found in a wide literature based on the presence of private information and noise (liquidity) investors (see, for example, Grossman and Stiglitz, 1980; DeLong et al., 1990). Then the difference-of-opinion approach by Varian (1985, 1989) and Harris and Raviv (1993) dispenses with the noise investors and obtains diverse posterior beliefs from the differences in the way investors interpret common information. A third method also exists to explain diverse posterior beliefs by relaxing the assumption of common prior, as in Harrison and Kreps (1978) and Morris (1996). Harrison and Kreps studied a model in which agents have different beliefs about the stochastic process of future dividends. They established the existence of the "minimal consistent price scheme" which is a partial equilibrium version of the Radner's "equilibrium of plans, prices and price expectations" (Radner, 1972). They also demonstrated the existence of positive speculative premiums, but only by providing some numerical examples. Morris (1996) adopted a simplified framework in which dividends follow an independently and identically distributed (i.i.d.) binomial distribution. The agents may have different prior distributions, but the difference of beliefs will disappear as agents learn from observing realized dividends. So the price of a risky asset can be greater than its fundamental value, but only initially, and the difference will converge to zero as investors' beliefs converge with Bayesian learning when more information becomes available. Such a framework may be more appropriate for modeling asset pricing during initial public offerings, but not for other speculative phenomena. Besides, the framework of i.i.d. binomial distribution for dividends is also quite limited.

According to Morris (1996), the result of Harrison and Kreps "has apparently been largely ignored, presumably because of the assumption of (unmodeled) hetero-

[1] Other related studies include Hirshleifer (1975), Kreps (1977), Milgrom and Stokey (1982), Leach (1991) and Detemple and Murphy (1994).

geneity of expectations." (Morris, 1996, p. 1112). We intend to correct the weakness of Harrison and Kreps (1978) by providing a theory to justify the continued presence of diverse beliefs in a general Markovian framework. Since the heterogeneity of beliefs will be sustained in our model, our theory is also different from that of Morris (1996). This paper proposes a new framework for the study of speculative phenomena without the shortcomings mentioned above, through the introduction of the rational beliefs of Kurz (1994) (see also Kurz and Schneider, 1996; Kurz and Wu, 1996; Kurz and Beltratti, 1997; Nielsen, 1996, for further development and applications). The theory of rational beliefs assumes that agents have ample data and that an empirical distribution exists which is commonly known to all agents. The theory then shows that the empirical distribution can be uniquely extended to a probability measure on an infinite sequence of observations and that relative to that measure, the process of observed variables is stationary. We call that probability measure the "stationary measure" of the dynamics. The stationary measure may be different from the original measure, possibly non-stationary, under which the data was generated, but the stationary measure is the common empirical knowledge on which all agents agree.

Investors have diverse beliefs which are "rational" in the sense of being compatible with observed data. In a stable environment the investors can learn only about the stationary measure of observed data. Although the stationary measures of investors' beliefs will become the same as those of the data with complete learning, these beliefs may stay non-stationary and diverse. In other words, the set of rational beliefs compatible with data (having the same stationary measure) can be quite large, including those non-stationary beliefs which may differ on the timing of some rare events (such as structural changes). Therefore, investors may disagree even when they are allowed to learn from a substantial number of observations (see Kurz, 1994). Our model provides a foundation for the continued presence of heterogeneous expectations in speculative trading, which was not discussed by Harrison and Kreps or Morris. The framework of rational beliefs enables us to study many interesting phenomena. In particular, we demonstrate in Section 3 of this paper the emergence of endogenous uncertainty (see Kurz and Wu, 1996; Huang and Wu, 1999) in a Rational Belief Equilibria (RBE) and the continued deviation of asset prices from the agents' valuation if obliged to hold on to the asset forever. We show that positive speculative premiums will persist in a Markovian model of speculative trading. We also demonstrate how the Rational Belief equilibrium (RBE) price may differ from the Rational Expectations Equilibrium (REE) price.

Before introducing rational beliefs in Section 3, we will discuss the basic model and analyze the properties of asset prices with a general Markovian belief system in Section 2. We demonstrate that the equilibrium asset prices can be determined on the basis of a "representative belief", which is constructed systematically from the heterogeneous beliefs of agents in the economy. The equilibrium asset prices with speculative trading were demonstrated by Harrison and Kreps (1978) to be no less than any investor's valuation. Morris (1996) showed that asset prices are strictly greater than any investor's valuation under certain conditions, but in a simplified framework of i.i.d. binomial distribution for dividends. Unlike Morris (1996), we adopt a Markovian framework to model dividends and agents' beliefs. In our framework of Markovian dividend processes and Markovian beliefs, we find the conditions for the emergence of positive speculative premiums by utilizing the technique of constructing a “representative belief” for the economy. In addition, we provide analytical results to show how the premium depends upon the extent of heterogeneity of beliefs, whereas Morris (1996) obtained results only from numerical simulation. All proofs are provided in the Appendix. Section 4 concludes.

2 Heterogeneous Markovian beliefs

We consider an economy with a finite number of types of investors ($i = 1, \cdots, K$), each type having different expectations about the future values of a risky asset. Following Harrison and Kreps (1978) and Morris (1996), we also assume that investors are risk neutral, that each type of investor has infinite collective wealth, and that no investor can sell the asset short. As discussed by Harrison and Kreps (1978, Section VI), such a model is a good approximation to a world of risk averse investors with finite wealth. As for the no short sales assumption, allowing some finite amount of short selling would not change the main results (see Morris, 1996, p. 1122) in this type of model.[2]

2.1 The basic model

All investors have access to the same information set, and future dividends $\{d_t\}$ of the risky asset are believed to follow a specific exogenous stochastic process. Harrison and Kreps (1978) allowed for a general functional dependence of dividends on the current information set while Morris (1996) considered the case of an i.i.d. dividend process. We adopt the assumption that the dividends follow a finite-state stationary Markov chain, which is similar to Section III of Harrison and Kreps. Suppose the transition matrix of investors' beliefs is an $S \times S$ matrix,

$$Q^i = \begin{bmatrix} q^i_{11} & q^i_{12} & \cdots & q^i_{1S} \\ \vdots & \vdots & \cdots & \vdots \\ q^i_{s1} & q^i_{s2} & \cdots & q^i_{sS} \\ \vdots & \vdots & \cdots & \vdots \\ q^i_{S1} & q^i_{S2} & \cdots & q^i_{SS} \end{bmatrix} = \begin{bmatrix} q^i_1 \\ \vdots \\ q^i_s \\ \vdots \\ q^i_S \end{bmatrix}, \tag{1}$$

where q^i_s is the sth row vector of Q^i, and $q^i_{ss'}$ represents the probability of state s' occurring in the next period, given that the current state is s, $s = 1, \cdots, S$. The elements of q^i_s should be between 0 and 1, and the sum of all elements is equal to one. The investors also believe that the dividends follow a Markov process. As in Kaldor or Harrison and Kreps, we say that investors exhibit “speculative trading

[2] The assumption of infinite wealth can be relaxed without much change to our results so long as the class of investors has enough wealth to purchase the security they have chosen. Harrison and Kreps (1978) also discussed the alternative of introducing a bond that makes unnecessary the infinite wealth assumption. For further development, please see Wu and Guo (2002).

behavior" if they are willing to pay more for the risky asset with a right to resell, than what they would pay if obliged to hold on to it forever.[3] Investors are willing to pay a "speculative premium" for the anticipated gains from speculative trading. That is, investors of type i will buy the asset when the price of the security in state s is lower than his willingness to pay in that state,

$$p_s \leq \gamma \mathrm{E}_s^i(\vec{p} + \vec{d}) = \gamma q_s^i(\vec{p} + \vec{d}) \ , s = 1, \cdots S,$$

where p_s represents the price of the asset in state s, $\vec{p} = (p_1, \cdots, p_s, \cdots, p_S)'$, $\vec{d} = (d_1, \cdots, d_s, \cdots, d_S)'$, $\mathrm{E}_s^i(\cdot)$ represents expectation of investors of type i when state s occurs, and γ denotes the common discount rate, $\gamma < 1$. As the investors purchase the asset in state s, the price p_s goes up until the market price is equal to the highest willingness to pay among investors.

We can define a stationary consistent price scheme p_s^* as follows:

$$p_s^* = \max_i \gamma \mathrm{E}_s^i(\vec{p^*} + \vec{d}) = \max_i \gamma q_s^i(\vec{p^*} + \vec{d}) \ , \quad s = 1, \cdots S \ . \tag{2}$$

Harrison and Kreps (1978) showed that there are many non-stationary consistent price schemes. In Proposition 3 we will demonstrate that there exists a unique stationary consistent price scheme, which is also the equilibrium price.

Let $\vec{p^i}$ be the expected present value of subjectively evaluated dividends to an investor of type i. With the risk neutrality assumption we can derive the "subjective valuation" $\vec{p^i}$ of an investor of type i if obliged to hold the asset forever: $\vec{p^i} = \gamma Q^i(\vec{p^i} + \vec{d})$. Such a "subjective valuation" will be used to define "speculative premium" later. By applying the following Lemma, $\vec{p^i}$ can be solved uniquely:

$$\vec{p^i} = (I - \gamma Q^i)^{-1} \gamma Q^i \vec{d}. \tag{3}$$

Lemma 1 (McKenzie, 1960). *For any $S \times S$ transition matrix Q and $\gamma < 1$, $I - \gamma Q$ is invertible.*

2.2 The case of two states

Before considering the general case of S states in the next subsection, we will illustrate some basic ideas in the framework of 2×2 Markov chains ($S = 2$). Suppose the belief of investors of type i is

$$Q^i = \begin{bmatrix} 1 - a^i & a^i \\ 1 - b^i & b^i \end{bmatrix}, \quad i = 1, \cdots, K \,, \tag{4}$$

where a^i, b^i are all between 0 and 1. Suppose there are two possible values of dividends $\vec{d} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$, with $d_2 > d_1$.

[3] It is difficult to find a satisfactory definition of speculation (see the discussion in Hart and Kreps, 1986). The definition adopted here follows the lead of Keynes (1936), Kaldor (1939), Feiger (1976), Harrison and Kreps (1978) and Hart and Kreps (1986). Our rational belief model introduces a new element into the concept of speculation in this literature by allowing agents to speculate that the beliefs of other agents may change in the future and choose their portfolios accordingly. We are grateful to the anonymous referee for helpful comments on this point.

When there is only one type of investor in the economy or when there are no disagreements among different types of investors ($Q^i = Q$, $\forall i$), the market equilibrium price can be represented by (3). However, when we have heterogeneous beliefs, the phenomenon of speculative trading occurs as shown in the following example.

Example 1 (Harrison and Kreps, 1978). As in Harrison and Kreps (1978), we assume $K = 2$, $\gamma = 0.75$, $d_1 = 0$, $d_2 = 1$,

$$Q^1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}, \; Q^2 = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}.$$

By applying equation (3) we can find

$$\vec{p^1} = \begin{bmatrix} \frac{4}{3} \\ \frac{11}{9} \end{bmatrix} \approx \begin{bmatrix} 1.33 \\ 1.22 \end{bmatrix}, \; \vec{p^2} = \begin{bmatrix} \frac{16}{11} \\ \frac{21}{11} \end{bmatrix} \approx \begin{bmatrix} 1.45 \\ 1.91 \end{bmatrix}.$$

With the infinite wealth and no short sale assumptions, it can be conjectured that investors of type 2 will hold the asset and the market price will be $\vec{p^2}$. However, investors of type 1 can "speculate" by buying the asset in $s = 1$ with the intention of selling it when $s = 2$ occurs. Such a speculative plan can generate a revenue of

$$[\frac{1}{2}(0.75) + (\frac{1}{2})^2(0.75)^2 + \cdots] \cdot (1 + \frac{21}{11}) \approx 1.75\,,$$

which is greater than the purchase cost of $\frac{16}{11}$ in $s = 1$. So the market price in $s = 1$ should be at least 1.75. That is, the market price should become $\begin{bmatrix} 1.75 \\ 1.91 \end{bmatrix}$, due to the speculative behavior of type 1 investors. However, 1.91 cannot be the final price in $s = 2$ since investors of type 2 can "speculate" by buying the asset in $s = 2$ with the intention of selling it when $s = 1$ occurs. Then there exists another speculation plan and so on. The speculation process will continue until it converges. The limit in this case is $\vec{p^*} \approx \begin{bmatrix} 1.85 \\ 2.08 \end{bmatrix}$.

Harrison and Kreps demonstrate that the infinite progression as in the above example finally stops and achieves a "minimal consistent price scheme", which is also the market price. In the following proposition, we obtain a characterization of the market price inspired by the treatment in Harrison and Kreps. We introduce a concept of "representative belief" which is constructed from the beliefs of all investors. Instead of getting an approximate limit, we can determine the market price precisely and efficiently on the basis of the "representative beliefs."

Proposition 1. *For the $S = 2$ case, there exists a unique (stationary) market price $\vec{p^*}$ and a representative belief Q^*, such that*

$$\vec{p^*} = (I - \gamma Q^*)^{-1} \gamma Q^* \vec{d}, \tag{5}$$

$$Q^* = \begin{bmatrix} 1 - \max_i a^i & \max_i a^i \\ 1 - \max_i b^i & \max_i b^i \end{bmatrix}. \tag{6}$$

We can apply this result to Example 1:

Example 1 (Extended). We can construct a "representative belief" Q^* by equation (6) of Proposition 1, and use equation (5) in Proposition 1 to derive $\vec{p^*}$:

$$Q^* = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}, \quad \vec{p^*} = \begin{bmatrix} \frac{24}{13} \\ \frac{27}{13} \end{bmatrix} \approx \begin{bmatrix} 1.85 \\ 2.08 \end{bmatrix},$$

which is the same as the "minimal consistent price scheme" derived by Harrison and Kreps. Note that our method efficiently generates quite a precise result (in fractions) in just one step, without having to go through an infinite sequence of finding various speculative plans.

Next we consider whether positive speculative premiums would exist in this economy. Speculative premiums are defined as the differences between the market price $\vec{p^*}$ and the subjective valuation $\vec{p^i}$ by investors of type i ($i = 1, \cdots, K$) if investors are obliged to hold the asset forever. Only when the market price is strictly greater than the subjective valuation of all investors, can we say that speculative premiums are positive. In Example 1, the speculative premiums $\vec{p^*} - \max\{\vec{p^1}, \vec{p^2}\}$ are positive; however, if there is a "dominant" investor j who has the highest valuation in all states, i.e., $a^j = \max_i a^i$ and $b^j = \max_i b^i$, this investor becomes the representative investor in the market with his belief Q^j being selected as Q^*. Then the speculative premium is zero. Morris (1996) demonstrated the existence of positive speculative premiums with an i.i.d. dividend process. In the following proposition, we provide conditions for the existence of positive speculative premiums in a Markovian framework, which was also adopted by Harrison and Kreps (1978) in their Section III. We also study how the size of premiums relates to the extent of heterogeneity of beliefs. Without loss of generality, we assume that $a^1 = \max_i a^i$, and $b^2 = \max_i b^i$ (no dominant investor).

Proposition 2. *Suppose there is no dominant investor ($a^1 = \max_i a^i$ and $b^2 = \max_i b^i$) in the market for the $S = 2$ case.*

(a) If $b^2 \neq 1$, then strictly positive speculative premiums exist in each state.

(b) The speculative premium $\vec{p^*} - \max\{\vec{p^1}, \vec{p^2}\} = \min\{\vec{p^*} - \vec{p^1}, \vec{p^*} - \vec{p^2}\}$, where

$$\vec{p^*} - \vec{p^1} = \frac{\gamma}{1-\gamma} \cdot \frac{1}{1+(a^1-b^2)\gamma} \cdot \frac{1}{1+(a^1-b^1)\gamma} \cdot (d_2 - d_1) \cdot (b^2 - b^1) \cdot \begin{bmatrix} a^1\gamma \\ 1-\gamma+a^1\gamma \end{bmatrix}, \tag{7}$$

$$\vec{p^*} - \vec{p^2} = \frac{\gamma}{1-\gamma} \cdot \frac{1}{1+(a^1-b^2)\gamma} \cdot \frac{1}{1+(a^2-b^2)\gamma} \cdot (d_2 - d_1) \cdot (a^1 - a^2) \cdot \begin{bmatrix} 1-b^2\gamma \\ (1-b^2)\gamma \end{bmatrix}. \tag{8}$$

(c) The size of speculative premiums increases as a^1 or b^2 increases and as a^2 or b^1 decreases:

$$\frac{\partial(\vec{p^*} - \vec{p^1})}{\partial b^1} < 0 < \frac{\partial(\vec{p^*} - \vec{p^1})}{\partial b^2}, \quad \frac{\partial(\vec{p^*} - \vec{p^2})}{\partial a^2} < 0 < \frac{\partial(\vec{p^*} - \vec{p^2})}{\partial a^1}. \tag{9}$$

This proposition demonstrates that strictly positive speculative premiums exist in a Markovian framework unless there is a dominant investor or the investors who hold the asset do not expect to sell it in the next period with probability 1, i.e., they do not expect to speculate at all. The later exception is described mathematically by the condition $b^2 \neq 1$. Since $a^1 - a^2$ and $b^2 - b^1$ measure the extent of heterogeneity of beliefs in the economy, changing a^i or b^i, while holding the other parameter constant in Proposition 2(c) tells us that the size of speculative premiums is an increasing function of the heterogeneity of beliefs.

2.3 The case of S states

Next we extend the results in the 2×2 Markov chains to the general case of $S \times S$ Markov chains. With Lemma 1, the subjective valuation by investors of type i can also be solved for the general $S \times S$ case,

$$\vec{p^i} = (I - \gamma Q^i)^{-1} \gamma Q^i \vec{d}. \tag{3}$$

Following the same reasoning as Proposition 1, we can find a type $i(s)$ of investors who have the highest valuation of the asset for any given state s. We will show that the market equilibrium price for the general $S \times S$ case can also be written as follows:

$$\vec{p^*} = \gamma \begin{bmatrix} q_1^{i(1)} \\ \cdot \\ q_s^{i(s)} \\ \cdot \\ q_S^{i(S)} \end{bmatrix} (\vec{p^*} + \vec{d}) = \gamma Q^* (\vec{p^*} + \vec{d}) \,,$$

where Q^* is the "representative belief" of a fictitious investor. In contrast to the 2×2 case with a unique Q^* as in Proposition 1, the representative belief is not necessarily unique in the general case. However, there exists a unique equilibrium price.

Example 2. In a 3×3 Markovian framework, there may exist multiple representative beliefs:

$$Q^1 = \begin{bmatrix} 0.5\ 0.4\ 0.1 \\ 0.5\ 0.4\ 0.1 \\ 0.3\ 0.3\ 0.4 \end{bmatrix}, Q^2 = \begin{bmatrix} 0.3\ 0.2\ 0.5 \\ 0.3\ 0.5\ 0.2 \\ 0.7\ 0.2\ 0.1 \end{bmatrix}, Q^3 = \begin{bmatrix} 0.4 & 0.4 & 0.2 \\ 0.4 & 0.5 & 0.1 \\ 0.22 & 0.43 & 0.35 \end{bmatrix}.$$

Given the dividend vector $\vec{d} = [0, 0.5, 1]'$ and discount factor $\gamma = 0.75$, there are two representative beliefs Q^{*1} and Q^{*2}, but a unique equilibrium price $\vec{p^*}$ still exists.

$$\vec{p^*} = \begin{bmatrix} 1.6544 \\ 1.5221 \\ 1.6103 \end{bmatrix}, Q^{*1} = \begin{bmatrix} 0.3\ 0.2\ 0.5 \\ 0.3\ 0.5\ 0.2 \\ 0.3\ 0.3\ 0.4 \end{bmatrix}, Q^{*2} = \begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0.3 & 0.5 & 0.2 \\ 0.22 & 0.43 & 0.35 \end{bmatrix}.$$

In fact, the third row of Q^{*1} is constructed from Q^1 and the third row of Q^{*2} is constructed from Q^3. When $s = 3$, both rows satisfy

$$p_3^* = \gamma q_3^i(\vec{p^*} + \vec{d}) = 1.6103\,,\ i = 1, 3.$$

Now we can study the general "representative belief" by considering the set of possible "combined beliefs" of some fictitious investor f, $f = 1, \cdots, K^S$:

$$\Psi = \{Q^f | q_s^f = q_s^i\,, i = 1, \cdots K, s = 1, \cdots, S\}\ .$$

The "representative belief" Q^* can be found as an element in Ψ. In Harrison and Kreps (1978, pp.331), the equilibrium price was calculated by iterated speculation of different investors until the price converges, as shown in Example 1. A shortcoming of this procedure is that it needs an infinite number of steps to find the solution, which is an approximation at best. Introducing "representative beliefs," as done in the last subsection, provides both a precise and efficient solution. Since the number of elements in Ψ is equal to K^S, just going through every element member of Ψ to find Q^* cannot be undertaken easily when the number of types of investors K becomes large (for example, where $K = 100$ and $S = 4$, it may need 100^4 steps). However, in what follows, we provide an efficient algorithm to discover equilibrium price $\vec{p^*}$ and representative belief Q^* in just a few steps. Furthermore, the number of steps depends only on the number of state S, not the number of types of investors K. Our experiences with numerical simulation suggest that, in a typical 4×4 case, it only needs at most 4 steps even when the number of types of investors is very large. The algorithm is as follows:

Step 1: Compute the subjective valuation $\vec{p^i} = (I - \gamma Q^i)^{-1}\gamma Q^i \vec{d}$ if investors of type i are obliged to hold forever, $i = 1, \cdots, K$. For each s, find the highest valuation $p_s^{i0(s)}$ and its corresponding type $i0(s)$. Set

$$Q^{f0} = \begin{bmatrix} q_1^{i0(1)} \\ \vdots \\ q_S^{i0(S)} \end{bmatrix} \in \Psi\ .$$

Step 2: Compute the corresponding price $\vec{p}^{f0} = (I - \gamma Q^{f0})^{-1}\gamma Q^{f0}\vec{d}$ for the fictitious belief Q^{f0} constructed in Step 1. Then compute the "willingness to pay" $\vec{W^{i0}} = \gamma Q^i(\vec{p}^{f0} + \vec{d})$ associated with $\vec{p}^{f0}$ for each type i. Find the highest willingness to pay for each state s and its corresponding type $i1(s)$. Set

$$Q^{f1} = \begin{bmatrix} q_1^{i1(1)} \\ \vdots \\ q_S^{i1(S)} \end{bmatrix} \in \Psi\ .$$

Let $Q^{f1} = F(Q^{f0})$, which defines a mapping $F:\ \Psi \to \Psi$.

Step 3: Compute the corresponding price $\vec{p}^{f1} = (I - \gamma Q^{f1})^{-1}\gamma Q^{f1}\vec{d}$, for the fictitious belief Q^{f1} constructed in Step 2. If $\vec{p}^{f1} = \vec{p}^{f0}$, stop the algorithm and list the price $\vec{p}^{f1}$ and belief Q^{f1} as the equilibrium values. If $\vec{p}^{f1} \neq \vec{p}^{f0}$, repeat Step 2 until $\vec{p}^{f(n+1)} = \vec{p}^{fn} (= \vec{p^*})$. The corresponding $Q^{f(n+1)}$ and Q^{fn} are exactly the "representative belief" Q^* discussed earlier. Note that while the equilibrium price is unique the representative beliefs are not.

This algorithm searches through the elements of Ψ for a candidate for the representative belief. A fixed point of the mapping F is shown to exist in Proposition 3. Before presenting our formal results, we can illustrate the procedure for finding $\vec{p^*}$ and Q^* in the following examples:

Example 2 (continued). Given the beliefs Q^i for $i = 1, 2, 3$, we first compute their subjective valuation if obliged to hold the asset forever:

$$\vec{p^1} = \begin{bmatrix} 0.9726 \\ 0.9726 \\ 1.2145 \end{bmatrix}, \vec{p^2} = \begin{bmatrix} 1.4069 \\ 1.3288 \\ 1.1762 \end{bmatrix}, \vec{p^3} = \begin{bmatrix} 1.2208 \\ 1.1690 \\ 1.3589 \end{bmatrix}.$$

We can find that $i0(1) = 2$, $i0(2) = 2$ and $i0(3) = 3$ are the types $i0(s)$ with highest valuation in state $s = 1, 2, 3$. Then we can construct Q^{f0} and compute $\vec{p}^{f0}$:

$$Q^{f0} = \begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0.3 & 0.5 & 0.2 \\ 0.22 & 0.43 & 0.35 \end{bmatrix}, \quad \vec{p}^{f0} = \begin{bmatrix} 1.6544 \\ 1.5221 \\ 1.6103 \end{bmatrix}.$$

In fact, this algorithm converges in one step: $\vec{p}^{f0} = \vec{p^*}$. In all examples, the algorithm converges in a small number of steps. In the proof of Proposition 3, we provide such an efficient and finite algorithm, which is in contrast to the infinite algorithm of Harrison and Kreps (1978).

Proposition 3. *For the $S \times S$ Markovian beliefs, there exists a finite algorithm to find a unique market price $\vec{p^*}$ and at least one fictitious investor whose "representative belief" Q^* satisfies the following equation:*

$$\vec{p^*} = (I - \gamma Q^*)^{-1}\gamma Q^*\vec{d}, \tag{10}$$

$$\text{and} \quad Q^* \in \Psi = \{Q^f | q_s^f = q_s^i, i = 1, \cdots K, s = 1, \cdots, S\}, \tag{11}$$

where Ψ is the set of possible "combined beliefs" of a fictitious investor.

In Proposition 3, we follow Proposition 2 of Harrison and Kreps (1978) and present an alternative way of demonstrating the existence and uniqueness of stationary equilibrium price $\vec{p^*}$ (the proof is in the appendix). In contrast to Harrison and Kreps (1978), the result is demonstrated by introducing "representative belief" in a specialized framework of Markovian beliefs. In doing so, we obtain a sharpened characterization of the properties of equilibrium prices, as summarized by equations (10) and (11). Note that Harrison and Kreps (1978) adopt a general framework and formulate an infinite algorithm to find the "minimal consistent price scheme" while there are many non-stationary equilibrium prices. In fact, they allow the price to be infinite and do not prove the uniqueness explicitly. In our specialized (Markovian) framework, we provide a different formulation to obtain by a finite algorithm a unique stationary equilibrium price, which is the same as their "minimal consistent price scheme".

Next we demonstrate that there are positive speculative premiums except in two extreme cases. The first is the presence of a "dominant" investor. An investor i is defined to be "dominant" if [4]

$$\vec{p^i} \geq \vec{p^j} \text{ and } \vec{p^i} \geq Q^j(\vec{p^i} + \vec{d}) \ , \ \forall \ j. \tag{12}$$

The second exceptional case is when investors with probability one do not expect to sell the security on any subsequent date. A general condition $g^i_{ss'} > 0, \forall \ i, \ s, \ s'$ is sufficient to ensure that this case will not occur. Both conditions will be stated in Proposition 4.

With the general Markov chains, it is possible to get multiple "representative beliefs" in equilibrium, as illustrated in Example 2. Hence it is not straightforward to generalize the measure of the extent of heterogeneity of Proposition 2. However, we can use the difference between equilibrium price and investors' "willingness to pay" associated with $\vec{p^*}$ to measure the heterogeneity in the economy. We can then show that the size of speculative premium is an increasing function of this heterogeneity measure of the economy. Define

$$m^i_s = p^*_s - \gamma q^i_s(\vec{p^*} + \vec{d}) = \gamma(q^*_s - q^i_s)(\vec{p^*} + \vec{d}) \, , \tag{13}$$

which is dependent on the difference of beliefs $q^*_s - q^i_s$ and is evaluated with respect to $\vec{p^*} + \vec{d}$. It measures how the subjective valuation of the asset $\gamma q^i_s(\vec{p^*} + \vec{d})$ by investors of type i deviates from the market valuation p^*_s. From equation (2) we know that $m^i_s \geq 0$. Define $\vec{m^i}$ to be $[m^i_1, \cdots, m^i_S]'$. We can then demonstrate the relationship between the size of speculative premiums and the extent of heterogeneity $\vec{m^i}$ in the following proposition.

Proposition 4. *Suppose that there is no dominant investor and that* $q^i_{ss'} > 0$, $\forall \ i, \ s, \ s'$. *Then speculative premium of state s, as denoted as* $\pi_s = p^*_s - \max_i\{p^i_s\}$, *must be positive. In addition, the speculative premium is an increasing function of the extent of heterogeneity measured by* $\vec{m^i}$, *that is,*

$$\vec{p^*} - \vec{p^i} = (I - \gamma Q^i)^{-1}\vec{m^i} \ . \tag{14}$$

[4] One might think that

$$q^i_s\vec{d} \geq q^j_s\vec{d} \, , \ \forall j, \ s \, ,$$

could be one possible definition for agent i to be a "dominant" investor since he is the most optimistic with regard to dividends. It is equivalent to (12) only in the case of 2 states. There exist counter-examples such that investor i is dominant ($\vec{p^*} = \vec{p^i}$) but is not the most optimistic with regard to dividends when S is greater than 2. We are grateful to the referee for helpful comments on the concept of the "dominant investor."

3 Endogenous uncertainty with rational Markovian beliefs

In this section we will study how endogenous uncertainty may emerge with speculative trading. We also provide reasons to explain why heterogeneous beliefs can persist in a Markovian framework with rational beliefs.

3.1 The meaning of endogenous uncertainty

In the previous section we found that positive speculative premiums exist with sufficiently diverse beliefs. In the analysis we need not require the dividends to be distinct in all S states. The states $s = 1, \cdots, S$ can be used to represent possible values of market prices, of which only some are affected by the exogenously given values of dividends. We say that "endogenous uncertainty" is present in a Markovian economy if the endogenously determined equilibrium prices are distinct even when the exogenous variables (dividends) take the same value (see Kurz and Wu, 1996; Huang and Wu, 1999, for a formal definition and general discussion).

Utilizing the technique of constructing a "representative belief" as in the previous section, we find conditions under which market equilibrium prices can be different even when dividends are the same. In the following example there are two possible values of dividends ($d_L = 0$, $d_H = 1$), but three distinct market equilibrium prices are present for $S = 3$. We define this phenomenon as the emergence of endogenous uncertainty, which can occur in a specialization of the model of Harrison and Kreps (1978).

Example 3. Suppose $S = 3$, $\gamma = 0.75$. The market equilibrium prices can be determined by the "representative beliefs" Q^* as in the last section. In equation (15) there are three distinct prices while there are only two prices in equation (16).

$$\vec{d} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, Q^* = \begin{bmatrix} 0.4 \; 0.1 \; 0.5 \\ 0.2 \; 0.3 \; 0.5 \\ 0.3 \; 0.3 \; 0.4 \end{bmatrix}, \vec{p^*} = \begin{bmatrix} 1.995 \\ 2.171 \\ 2.089 \end{bmatrix}. \tag{15}$$

$$\vec{d} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, Q^* = \begin{bmatrix} 0.4 \; 0.1 \; 0.5 \\ 0.3 \; 0.2 \; 0.5 \\ 0.3 \; 0.3 \; 0.4 \end{bmatrix}, \vec{p^*} = \begin{bmatrix} 1.946 \\ 2.027 \\ 2.027 \end{bmatrix}. \tag{16}$$

A closer examination reveals one possible reason for the differences between equation (15) and (16). In (15), the probability $Pr^*(d_{s^{t+1}} = 1 | s^t = s)$ of getting high dividend ($d_H = 1$) given the current state s, according to the representative belief Q^*, is equal to $0.3 + 0.5 = 0.8$ when $s = 2$ and is equal to $0.3 + 0.4 = 0.7$ when $s = 3$. These two conditional probabilities are different. In (16), the two conditional probabilities given $s = 2$ and $s = 3$ are both equal to 0.7.

Example 3 represents a general phenomenon not only valid for representative belief Q^*, but also true for some given belief Q. In the following lemma, we state the conditions for the subjective valuation $\vec{p}^{\,Q}$ of an investors with belief Q to be the same in all states where the dividends are the same, i.e.,

$$\text{if } d_s = d_{s'}, \text{ then } p_s^Q = p_{s'}^Q .$$

Note that such an investor can be a fictitious investors as described in Proposition 3.

Lemma 2. *Given S states and 2 possible values of dividends ($d = d_L, d_H$), the necessary and sufficient condition for the subjective price $\vec{p}^{\,Q}$ of an investor with belief Q to be the same in all states where the dividends are the same is that the conditional probabilities of getting high (low) dividends are the same for all current states with high (low) dividends. This condition can be written as*

$$Pr^Q(d_{s^{t+1}} = d_H | s^t) = k_H \text{ , for all } s^t \text{ such that } d_{s^t} = d_H \text{ ,} \tag{17a}$$

and

$$Pr^Q(d_{s^{t+1}} = d_L | s^t) = k_L \text{ , for all } s^t \text{ such that } d_{s^t} = d_L \text{ .} \tag{17b}$$

Lemma 2 applies to the case of two possible values of dividends ($d_t \in D = \{d_L,\, d_H\}$), which is assumed in this section. We also assume that the stationary measure of the dividend process follows a Markov chain which can be represented by a transition matrix of the following form as in Kurz and Schneider (1996) and Kurz (1998):

$$\Lambda = \begin{bmatrix} \phi & 1-\phi \\ 1-\phi & \phi \end{bmatrix} . \tag{18}$$

Consider the rational expectations equilibrium (REE) approach in economic modeling where all agents are assumed to be able to carry out the necessary calculations to deduce the equilibrium price map $p_t = P(s_t)$ given the knowledge of the exogenous state s_t. The agents are then said to have "structural knowledge."[5] If all investors possess structural knowledge including equation (18), the investors will not choose any belief other than the one in equation (18). Then from Proposition 1 the REE price should be represented by equation (19). We will use the REE price $\vec{p^{\Lambda}}$ as a reference point for comparison later.

$$\begin{aligned} \vec{p^{\Lambda}} &= \begin{bmatrix} p_L^{\Lambda} \\ p_H^{\Lambda} \end{bmatrix} \\ &= \frac{\gamma}{(1-\gamma)(1+\gamma-2\gamma\phi)} \cdot \begin{bmatrix} (\gamma - 2\gamma\phi + \phi)d_L + (1-\phi)d_H \\ (1-\phi)d_L + (\gamma - 2\gamma\phi + \phi)d_H \end{bmatrix} . \end{aligned} \tag{19}$$

However, investors have neither structural knowledge nor information about the beliefs of other investors in reality. Therefore, they may form different subjective beliefs about the process of dividends and prices. As the equilibrium price depends on the state of beliefs y_t of all agents, so

$$p_t = P(s_t, y_t) \text{ .}$$

[5] One could interpret structural knowledge as common knowledge by all agents of the objectives of all investors and the law of motion for $\{d_t\}$. In general, structural knowledge also includes precise information of demand or supply functions and probability laws, which are usually unobservable.

Note that p_t now fluctuates when the state of beliefs varies. The component of economic fluctuations due to the agents' beliefs hence represents an important kind of uncertainty faced by all agents. This is referred to as "endogenous uncertainty" by Kurz and Wu (1996). In the rest of this paper we will study the emergence of endogenous uncertainty in a rational belief framework.

3.2 The rational belief framework

We now introduce the framework of rational beliefs of Kurz (1994) where agents do not have structural knowledge of the economy. If agents can form expectations without restrictions, we are in the framework of temporary equilibrium, which is criticized for its lack of rational utilization of information in learning. We impose a set of rationality conditions to place greater restrictions on the system of heterogeneous Markovian beliefs we have analyzed so far. Agents are said to have "rational beliefs" if their beliefs cannot be refuted by data. In a non-stationary environment, agents can learn only about the stationary measure of the observed data. Agents agree only on the stationary measure of the environment, but they can still disagree on the likelihood of some important and rare events or on the timing of regime changes even after exhausting all possibilities of learning.

Similarly to the overlapping generation framework adopted by Kurz (1998), we think of each investor as a "dynasty" consisting of a sequence of short-lived decision making agents. All investors live for a short finite life and those who belong to the same dynasty are linked by bequest motives. The theory of rational beliefs is based on the observation that our economy is a non-stationary environment with a sequence of changing regimes and the length of decision life of any agent is relatively short. During his economic life each agent has too few observations to determine if his private theory is correct or not. The private theory of each agent will be represented by an "assessment variable." The little learning which can be done during the lifetime of the agents cannot be used to refute agents' private theories or change their "assessment variables." This provides a theoretical foundation for the continued presence of heterogeneous expectation even when learning is allowed. In the rest of this paper we will study the properties of rational belief equilibria (RBE).

We assume that there are two types $i = 1, 2$ and many investors of each type.[6] The individual investor n of type i adopts a "state of belief" or an "assessment variable" $y^{i,n} \in Y^i = \{0, 1\}$, which is an i.i.d. random variable representing an investor's private signal or state of mind. Investors of type i receives at date t a random signal $y_t^{i,n}$ which determines his one-period ahead conditional probability as represented by a Markov matrix $Q_t^{i,n}$. They adopt matrix $Q_t^{i,0}$ as their belief when $y_t^{i,n} = 0$, and matrix $Q_t^{i,1}$ as their belief when $y_t^{i,n} = 1$. The probabilities of assessment variables for each agent are represented by Prob$\{y_t^{i,n} = 0\} = \alpha^i$ and Prob$\{y_t^{i,n} = 1\} = 1 - \alpha^i$. The vector $y = (y^{1,1}, \cdots, y^{1,N}, y^{2,1}, \cdots, y^{2,N})$ is a collection of individual states in an economy with $2N$ agents ($i = 1, 2$, $n = 1, \cdots, N$). Agents of type i who have the same assessment will have the same

[6] The results in this section hold for any finite number of types.

demand behavior. However, each agent knows his own assessment variables but not the assessment variables of any of the other agents in the economy, past or present. Within our framework, it is not possible to learn about others' private theories. It is also not possible to assess whether his private theory is wrong since the length of each economic regime is short and the agent does not have enough data.

Following Cass, Chichilnisky and Wu (1996) and Kurz (1998), we can define a "social state" to include all those collections of individual states which yield the same aggregate composition of investors in the economy. For example, all collections of individual states with half of type i investors being optimistic ($y^{i,n} = 1$) and half of type i investors being pessimistic ($y^{i,n} = 0$) yield the same aggregate composition of investors, so they are all represented by one social state for type i, denoted by its distribution $F^i = (0.5, 0.5)$. To be more precise, the distribution of type i investors can be represented by $F^i = (f^i, 1 - f^i)$, $0 < f^i < 1$, where f^i is the proportion of type i investors with assessment variables taking the value $y^{i,n} = 1$ (the rest $1 - f^i$ have assessment variables with the value $y^{i,n} = 0$). Let $\mathcal{F}^i$ be the space of possible distributions of type i investors, then distributions $F^i \in \mathcal{F}^i$.

Note that such a space of distributions $\mathcal{F}^i$ allows for possible correlation within the type. For example, $\mathcal{F}^i = \{(0.5, 0.5)\}$ corresponds to the case when individual assessments of type i investors are independently distributed with $\alpha^i = 0.5$. When individual assessments are perfectly correlated for all agents of type i, the possible distributions for type i investors can be represented by $\mathcal{F}^i = \{(0, 1),\ (1, 0)\}$, with two element. Other forms of correlation result in a non-degenerate representation of $\mathcal{F}^i$. For example, the element $F^i = (0.8, 0.2)$ of $\mathcal{F}^i = \{(0.8, 0.2), (0.2, 0.8)\}$ represents the case with a proportion 0.8 of type i investors adopting belief $Q^{i,1}$ and a proportion 0.2 of type i investors adopting belief $Q^{i,0}$, and similarly for $F^i = (0.2, 0.8)$. Although each investor of type i still adopts the belief $Q^{i,0}$ with probability α^i, for $F^i = (0.8, 0.2)$, 80% of them act together in choosing belief $Q^{i,1}$ while the rest of them act together in choosing the other belief $Q^{i,0}$. As for $F^i = (0.2, 0.8)$, 80% of them now choose $Q^{i,0}$ together and the rest choose $Q^{i,1}$ together.[7] This gives rise to the space of distributions $\mathcal{F}^i = \{(0.8, 0.2), (0.2, 0.8)\}$. So it must have two elements. It is a case of partial correlation.

Let there be J elements of $\mathcal{F}^i$ and $D = \{d_L, d_H\}$, then the state space considered in this section is $\hat{\mathcal{S}} = D \times \mathcal{F}^1 \times \mathcal{F}^2$ with $S = 2 \cdot J^2$ elements (two values for dividends, and two types of agents are assumed). Each of the S elements will be called a "social state" for the whole economy.

The stationary measure on $\hat{\mathcal{S}} = D \times \mathcal{F}^1 \times \mathcal{F}^2$ can be represented by the following transition matrix

$$\Gamma = \begin{bmatrix} \phi A & (1-\phi)A \\ (1-\phi)A & \phi A \end{bmatrix}, \tag{20}$$

[7] Since we adopt the assumptions of infinite wealth and risk-neutrality as in Harrison and Kreps (1978), only the optimists affect the equilibrium. The number of investors who are optimists or pessimists does not matter for the equilibrium price. To incorpoate a genuine influence of the distribution of investors, we need to study a model with finite wealth, as in Wu and Guo (2002).

where A is a $(J^2 \times J^2)$ transition matrix and the marginals of Γ on the dividend states D are represented by equation (18). We can represent the stationary transition matrix and belief matrices as

$$\Gamma = \begin{bmatrix} \Gamma_1 \\ \vdots \\ \Gamma_s \\ \vdots \\ \Gamma_S \end{bmatrix} = \begin{bmatrix} \Gamma_{11} & \cdots & \Gamma_{1S} \\ \vdots & \ddots & \vdots \\ \Gamma_{s1} & \cdots & \Gamma_{sS} \\ \vdots & \ddots & \vdots \\ \Gamma_{S1} & \cdots & \Gamma_{SS} \end{bmatrix}, \tag{21}$$

$$Q^{i,j} = \begin{bmatrix} q_1^{i,j} \\ \vdots \\ q_s^{i,j} \\ \vdots \\ q_S^{i,j} \end{bmatrix} = \begin{bmatrix} q_{11}^{i,j} & \cdots & q_{1S}^{i,j} \\ \vdots & \ddots & \vdots \\ q_{s1}^{i,j} & \cdots & q_{sS}^{i,j} \\ \vdots & \ddots & \vdots \\ q_{S1}^{i,j} & \cdots & q_{SS}^{i,j} \end{bmatrix}, \quad \begin{array}{l} i = 1, 2, \\ j = 0, 1. \end{array}$$

The stationary measure Γ is computed from data, and the investors' beliefs will be made consistent with Γ. Then the rationality constraint of Kurz and Schneider (1996) in a Markovian framework requires that for each individual investor

$$\alpha^i Q^{i,0} + (1 - \alpha^i) Q^{i,1} = \Gamma \text{ for } i = 1, 2\,, \tag{22}$$

where $\alpha^i = \text{Prob}\{y^{i,n} = 0\}$ and $1 - \alpha^i = \text{Prob}\{y^{i,n} = 1\}$. Note that in the rationality restriction we only use the ex-ante probability of private signals α^i, which may be different from the actual proportion f^i in the social state F^i. The equilibrium achieved in such a framework with a state space described above and beliefs satisfying (22) is defined to be a "Rational Belief Equilibrium (RBE) with social states of beliefs" (first introduced by Kurz, 1998).

Before proceeding with our analysis, we consider the benchmark case of rational expectation equilibrium (REE) when all investors' beliefs coincide with the stationary measure, i.e. $Q^{i,j} = \Gamma$, $\forall i, j$. One can show that there exists no endogenous uncertainty and there are only two possible values of prices. The market equilibrium price vector achieved in this benchmark case is reduced to $\vec{p^\Lambda}$, which is an REE price represented by equation (19). We can now write $\vec{p^\Lambda} = \left[p_L^\Lambda, p_L^\Lambda, p_L^\Lambda, p_L^\Lambda,\ p_H^\Lambda, p_H^\Lambda, p_H^\Lambda, p_H^\Lambda\right]'$. Note that the equilibrium price $\vec{p^\Lambda}$ has only two possible values p_L^Λ, p_H^Λ, associated with dividends $d_s = d_L, d_H$, respectively.

With a rational belief structure as specified by (22), investors may use different beliefs from time to time. Let the current belief at date t be $Q_t^{i,j}$ by an investor of type i with $y^{i,n} = j$, $j = 0, 1$. The agent knows his current belief $Q_t^{i,j}$, but before receiving private signals $y_{t+1}^{i,n}$ he does not know his future beliefs $Q_{t+1}^{i,j}$ except that it is one of the two Markovian matrices in equation (22) with probabilities α^i and $1-\alpha^i$. The expectation at date t of his future beliefs at date $t+1$ and beyond is thus equal to the weighted average of these two Markovian matrices, which is equal to Γ as required by (22). We can show in the following lemma that the expected revenue from sales of the asset in the next period is no less than the expected revenue from sales of the asset in a later period. (the right hand side of equation (23)):

Lemma 3. *For any investor of type i the following holds in an RBE:*

$$\gamma Q^{i,j}(\vec{p^*} + \vec{d}) \geq \gamma Q^{i,j}(\gamma \Gamma(\vec{p^*} + \vec{d}) + \vec{d}) \tag{23}$$

From Lemma 3, it is clear that selling in later periods will not provide any higher revenues than selling in the next period, as shown by the following inequalities:

$$\begin{aligned} &\gamma Q^{i,j}(\vec{p^*} + \vec{d}) \\ &\geq \gamma Q^{i,j}(\gamma \Gamma(\vec{p^*} + \vec{d}) + \vec{d}) \\ &\geq \gamma Q^{i,j}(\gamma \Gamma(\gamma \Gamma(\vec{p^*} + \vec{d}) + \vec{d}) + \vec{d}) \end{aligned} \tag{24}$$

Hence, in equilibrium the valuation of any investor can be represented just by his expected revenue from sales in the next period. With this lemma, the rational belief framework developed here can accommodate for the model of Harrison and Kreps (1978).

Lemma 3 performs an important role in the dynamic decision of speculation. As investors of type i adopt beliefs of $Q^{i,0}$, $Q^{i,1}$, Lemma 3 allows us only to consider simple trading strategies consisting of selling the security in the next period instead of selling in later periods. So we can apply the technique and results developed in the previous sections even when the beliefs adopted by investors are not constant over time.

3.3 The properties of rational belief equilibrium

Now we can show that endogenous uncertainty may exist in RBE, using the technique of constructing a "representative belief", developed in the previous sections. We also demonstrate that the RBE prices can deviate from the REE prices $\vec{p^\Lambda}$ of equation (19). The following example illustrates the possibility of the presence of endogenous uncertainty. Agents may not have structural knowledge, but they still try to learn from the data until the stationary measures of their beliefs coincide with Γ, i.e., their beliefs satisfy the rationality restrictions of the theory of rational beliefs.

Example 4. Suppose that $D = \{1, 2\}$ and each $\mathcal{F}^i = \{(0.8, 0.2),\ (0.2, 0.8)\}$ has two elements. The state space is $\hat{\mathcal{S}} = D \times \mathcal{F}^1 \times \mathcal{F}^2$ with eight elements. To construct a Γ consistent with equation (20), let A be a 4×4 matrix with all elements equal to 0.25, $\phi = 0.5$ and Γ be a 8×8 matrix with all elements equal to 0.125. Let $\alpha^1 = \alpha^2 = 0.5$, $\gamma = 0.75$ and B be a 1×4 matrix with all elements equal to 0.25. This is the case of i.i.d. type states with no correlation for the stationary distribution. Suppose

$$Q^{1,0} = \begin{bmatrix} 0.30B & 0.70B \\ 0.20B & 0.80B \\ 0.40B & 0.60B \\ 0.10B & 0.90B \\ 0.60B & 0.40B \\ 0.55B & 0.45B \\ 0.68B & 0.32B \\ 0.58B & 0.42B \end{bmatrix}, \; Q^{1,1} = \begin{bmatrix} 0.70B & 0.30B \\ 0.80B & 0.20B \\ 0.60B & 0.40B \\ 0.90B & 0.10B \\ 0.40B & 0.60B \\ 0.45B & 0.55B \\ 0.32B & 0.68B \\ 0.42B & 0.58B \end{bmatrix},$$

$$Q^{2,0} = \begin{bmatrix} 0.60B & 0.40B \\ 0.80B & 0.20B \\ 0.55B & 0.45B \\ 0.60B & 0.40B \\ 0.25B & 0.75B \\ 0.35B & 0.65B \\ 0.40B & 0.60B \\ 0.45B & 0.55B \end{bmatrix}, \; Q^{2,1} = \begin{bmatrix} 0.40B & 0.60B \\ 0.20B & 0.80B \\ 0.45B & 0.55B \\ 0.40B & 0.60B \\ 0.75B & 0.25B \\ 0.65B & 0.35B \\ 0.60B & 0.40B \\ 0.55B & 0.45B \end{bmatrix}.$$

we can check that these belief matrices satisfy the rationality restrictions (22) since $\alpha^1 = 0.5$ and $\alpha^2 = 0.5$: $\frac{1}{2}Q^{1,0} + \frac{1}{2}Q^{1,1} = \Gamma$, $\frac{1}{2}Q^{2,0} + \frac{1}{2}Q^{2,1} = \Gamma$. At any of the eight social states, all types of beliefs $\{Q^{1,1}, Q^{1,0}, Q^{2,1}, Q^{2,0}\}$ are present. Utilizing the techniques of constructing "representative beliefs" of Proposition 3, we can find $\vec{p^*}$ and Q^* by equation (10) and the algorithm presented in Section 2.3:

$$Q^* = \begin{bmatrix} 0.30B & 0.70B \\ 0.20B & 0.80B \\ 0.40B & 0.60B \\ 0.10B & 0.90B \\ 0.25B & 0.75B \\ 0.35B & 0.65B \\ 0.32B & 0.68B \\ 0.42B & 0.58B \end{bmatrix},$$

$$\vec{p^*} = \begin{bmatrix} 5.0799 \\ 5.1504 \\ 5.0094 \\ 5.2209 \\ 5.1152 \\ 5.0447 \\ 5.0658 \\ 4.9953 \end{bmatrix} > \vec{p^\Lambda} = \begin{bmatrix} 4.5 \\ 4.5 \\ 4.5 \\ 4.5 \\ 4.5 \\ 4.5 \\ 4.5 \\ 4.5 \end{bmatrix}, \quad \vec{p^*} - \max_i\{\vec{p^i}\} = \begin{bmatrix} 0.3300 \\ 0.3412 \\ 0.3188 \\ 0.3524 \\ 0.3789 \\ 0.4029 \\ 0.4713 \\ 0.4115 \end{bmatrix}.$$

Note that in Example 4 speculative premiums $p_s^* - \max_i\{p_s^i\}$ are positive since the conditions of Proposition 4 are satisfied. Since the results of Proposition 4 also apply in the framework of rational beliefs, the focus of this section will go beyond finding conditions for positive speculative premiums. In Example 4, also note that the RBE prices $\vec{p^*}$ exhibit the phenomena of "endogenous uncertainty". The agents in the economy form their beliefs conditional on the social states and may end up in equilibrium with distinct prices for different social states. However, there are cases in which there exists RBE without endogenous uncertainty. In the following proposition, we demonstrate the generic presence of endogenous uncertainty in a Rational Belief Equilibrium with social states of beliefs.

In the proof of Proposition 5, we employ that fact that the necessary and sufficient conditions for the nonexistence of endogenous uncertainty for RBE are

$$\sum_{k=1}^{\hat{s}} q^*_{sk} = k_L, \text{ for } s\text{=1 to } \hat{s}\,, \tag{25}$$

$$\sum_{k=\hat{s}+1}^{S} q^*_{sk} = k_H, \text{ for } s = \hat{s}+1 \text{ to } S\,, \tag{26}$$

where $s = 1, \cdots \hat{s}$ correspond to the case of low dividend, $s = \hat{s}+1, \cdots S$ correspond to the case of high dividend, $\hat{s} = \frac{S}{2}$ and Q^* with elements q^*_{sk} is the representative belief. When these conditions are not satisfied, there exists endogenous uncertainty. We identify the space of possible rational beliefs of all investors as $\mathcal{Q}$ with

$$\vec{Q} = \{Q^{1,0}, Q^{1,1}, \cdots, Q^{I,0}, Q^{I,1}\} \in \mathcal{Q} \subset \Delta^{2SI},$$
$$Q^{i,j} \in \Delta^S = \{q^{i,j}_s \in R^S \mid \sum_{s'} q^{i,j}_{ss'} = 1, 0 \le q^{i,j}_{ss'} \le 1\}$$
$$\text{and } \alpha^i Q^{i,0} + (1-\alpha^i) Q^{i,1} = \Gamma\ .$$

Given a uniform measure on $\mathcal{Q}$, we can show that there exists generally (on $\mathcal{Q}$) RBE with endogenous uncertainty. Note that $(Q^{i,0}, Q^{i,1}) \in \Delta^{2S}$ implies that dim(Δ^{2S}) $= 2S-2$. With S rationality constraints for each i, rational beliefs $(Q^{i,0}, Q^{i,1})$ is in the space of dimension $(2S-2)-S = S-2$. Hence the space of rational beliefs of all agents $\mathcal{Q}$ has a dimension of $(S-2)I$. From Proposition 3, we can represent the equilibrium price $\vec{p^*}$ by some representative belief Q^*. Then we apply Lemma 2 to such a Q^* to obtain the result in Proposition 5(b):

Proposition 5. *(a). Given the beliefs of agents there is a unique RBE with social states of beliefs.*
(b). There exists generically RBE with endogenous uncertainty. Such an RBE price $\vec{p^*}$ will not be equal to the REE price $\vec{p^\Lambda}$.
(c). When there is no endogenous uncertainty, the RBE price $\vec{p^*}$ may not be equal to the REE price $\vec{p^\Lambda}$. The RBE price $\vec{p^*}$ is equal to the REE price $\vec{p^\Lambda}$ if, and only if, $k_L = \phi$ and $k_H = \phi$ hold, in addition to equations (25) and (26).

Note that in Harrison and Kreps (1978) even when agents have diverse beliefs, $\vec{p^\Lambda}$ could still be the equilibrium price, for instance, when a dominant investor has rational expectations. However, under rational beliefs this is only possible when all agents have rational expectations. This is another way, in which the rational beliefs case differentiates itself from the Harrison and Kreps model.

The other interesting phenomenon (besides endogenous uncertainty) demonstrated in Example 4 is that the RBE price $\vec{p^*}$ is higher than the REE price $\vec{p^\Lambda}$. In fact, we can show later that the RBE price $\vec{p^*}$ can be higher than $\vec{p^\Lambda}$ as long as the beliefs of agents remain sufficiently diverse, which will be the focus of the rest of this section. However, the equilibrium price may be lower than $\vec{p^\Lambda}$ of REE if the agents' beliefs become perfectly correlated. We will first discuss the case of perfect correlation, then present the results without perfect correlation in Proposition 6.

If there are perfect correlations among the individual assessment variables $y^{i,n}$ within type i, then $y^{i,n} = y^i \in Y^i = \{0,1\}$, $\mathcal{F}^i = \{(0,1), (1,0)\}$ for all i. We refer to the equilibrium of this case as "RBE with perfect correlation within types." The state space can be represented by the following index mapping Φ, with $s = 1, \cdots, 8$:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} = \Phi \begin{bmatrix} d = d_L \; y^1 = 0 \; y^2 = 0 \\ d = d_L \; y^1 = 0 \; y^2 = 1 \\ d = d_L \; y^1 = 1 \; y^2 = 0 \\ d = d_L \; y^1 = 1 \; y^2 = 1 \\ d = d_H \; y^1 = 0 \; y^2 = 0 \\ d = d_H \; y^1 = 0 \; y^2 = 1 \\ d = d_H \; y^1 = 1 \; y^2 = 0 \\ d = d_H \; y^1 = 1 \; y^2 = 1 \end{bmatrix} . \tag{27}$$

The structure of the economy as represented by the above equation is not known to investors in the economy, but the stationary measure Γ on $\hat{\mathcal{S}} = D \times \mathcal{F}^1 \times \mathcal{F}^2$ can be learned by agents:

$$\Gamma = \begin{bmatrix} \phi A & (1-\phi)A \\ (1-\phi)A & \phi A \end{bmatrix}, \tag{20}$$

where

$$A = [A_{ij}] = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} = \begin{bmatrix} a_1 \; \alpha^1 - a_1 \; \alpha^2 - a_1 \; 1 + a_1 - \alpha^1 - \alpha^2 \\ a_2 \; \alpha^1 - a_2 \; \alpha^2 - a_2 \; 1 + a_2 - \alpha^1 - \alpha^2 \\ a_3 \; \alpha^1 - a_3 \; \alpha^2 - a_3 \; 1 + a_3 - \alpha^1 - \alpha^2 \\ a_4 \; \alpha^1 - a_4 \; \alpha^2 - a_4 \; 1 + a_4 - \alpha^1 - \alpha^2 \end{bmatrix} \tag{28}$$

is a 4×4 transition matrix in which a_k, k=1,2,3,4, measure correlation across types. Note that (28) implies that Prob$\{y^i = 0\} = \alpha^i$ for $i = 1, 2$, which is compatible with our specification for individual assessment variables. Type i investors adopt beliefs $Q^{i,0}$ with an ex-ante probability α^i and $Q^{i,1}$ with ex-ante probability of $1 - \alpha^i$. In general, investors of type i adopt one of the two beliefs independently, depending on their assessment variable y^i, so that both beliefs are present in the economy. However, with perfect correlation as described by equation (27), there is only one possible type of belief for each type of investors at any state. For example, at state $s = 1$, the investors of type 1 all have belief $Q^{1,0}$ ($y^1 = 0$) and investors of type 2 all have belief $Q^{2,0}$ ($y^2 = 0$). In the following examples, we show that the RBE price with perfect correlation within types can be either higher or lower than REE price $\vec{p^\Lambda}$.

Example 5. Given Γ, γ, A and B as specified in Example 4. Suppose

$$Q^{1,0} = \begin{bmatrix} 0.40B & 0.60B \\ 0.55B & 0.45B \\ 0.40B & 0.60B \\ 0.80B & 0.20B \\ 0.60B & 0.40B \\ 0.15B & 0.85B \\ 0.75B & 0.25B \\ 0.30B & 0.70B \end{bmatrix}, \; Q^{1,1} = \begin{bmatrix} 0.60B & 0.40B \\ 0.45B & 0.55B \\ 0.60B & 0.40B \\ 0.20B & 0.80B \\ 0.40B & 0.60B \\ 0.85B & 0.15B \\ 0.25B & 0.75B \\ 0.70B & 0.30B \end{bmatrix},$$

$$Q^{2,0} = \begin{bmatrix} 0.60B & 0.40B \\ 0.65B & 0.35B \\ 0.45B & 0.55B \\ 0.30B & 0.70B \\ 0.30B & 0.70B \\ 0.45B & 0.55B \\ 0.65B & 0.35B \\ 0.90B & 0.10B \end{bmatrix}, \; Q^{2,1} = \begin{bmatrix} 0.40B & 0.60B \\ 0.35B & 0.65B \\ 0.55B & 0.45B \\ 0.70B & 0.30B \\ 0.70B & 0.30B \\ 0.55B & 0.45B \\ 0.35B & 0.65B \\ 0.10B & 0.90B \end{bmatrix}.$$

Given perfect correlation as described by equation (27), at state $s = 1$ only the first row of $Q^{1,0}$ will become "effective" since state $s = 1$ is described by $y^1 = 0$. Hence we can construct the beliefs $Q^{i,e}$ of these two types of agents, $i = 1, 2$, that become effective for states $s = 1, \cdots, 8$ and the corresponding "representative beliefs" Q^*:

$$Q^{1,e} = \begin{bmatrix} 0.40B & 0.60B \\ 0.55B & 0.45B \\ 0.60B & 0.40B \\ 0.20B & 0.80B \\ 0.60B & 0.40B \\ 0.15B & 0.85B \\ 0.25B & 0.75B \\ 0.70B & 0.30B \end{bmatrix}, \; Q^{2,e} = \begin{bmatrix} 0.60B & 0.40B \\ 0.35B & 0.65B \\ 0.45B & 0.55B \\ 0.70B & 0.30B \\ 0.30B & 0.70B \\ 0.55B & 0.45B \\ 0.65B & 0.35B \\ 0.10B & 0.90B \end{bmatrix}, \; Q^* = \begin{bmatrix} 0.40B & 0.60B \\ 0.35B & 0.65B \\ 0.45B & 0.55B \\ 0.20B & 0.80B \\ 0.30B & 0.70B \\ 0.15B & 0.85B \\ 0.25B & 0.75B \\ 0.10B & 0.90B \end{bmatrix}.$$

Applying the technique developed in the last section, we can find equilibrium price $\vec{p^*} > \vec{p^\Lambda}$.

$$\vec{p^*} = \begin{bmatrix} 5.1549 \\ 5.1972 \\ 5.1127 \\ 5.3239 \\ 5.2394 \\ 5.3662 \\ 5.2817 \\ 5.4085 \end{bmatrix} > \vec{p^\Lambda} = \begin{bmatrix} 4.5 \\ 4.5 \\ 4.5 \\ 4.5 \\ 4.5 \\ 4.5 \\ 4.5 \\ 4.5 \end{bmatrix}.$$

Note that the RBE price $\vec{p^*}$ demonstrates endogenous uncertainty and $p_s^* > p_s^\Lambda$ in all states. We can also find cases in which the RBE prices p_s^* can be lower than REE price p_s^Λ in all states, as in the following example.

Example 6. Given Γ, γ, A and B as specified in Example 4. Suppose

$$Q^{1,0} = \begin{bmatrix} 0.50B & 0.50B \\ 0.80B & 0.20B \\ 0.20B & 0.80B \\ 0.10B & 0.90B \\ 0.80B & 0.20B \\ 0.65B & 0.35B \\ 0.45B & 0.55B \\ 0.20B & 0.80B \end{bmatrix}, \; Q^{1,1} = \begin{bmatrix} 0.50B & 0.50B \\ 0.20B & 0.80B \\ 0.80B & 0.20B \\ 0.90B & 0.10B \\ 0.20B & 0.80B \\ 0.35B & 0.65B \\ 0.55B & 0.45B \\ 0.80B & 0.20B \end{bmatrix},$$

$$Q^{2,0} = \begin{bmatrix} 0.80B & 0.20B \\ 0.40B & 0.60B \\ 0.70B & 0.30B \\ 0.20B & 0.80B \\ 0.75B & 0.25B \\ 0.20B & 0.80B \\ 0.80B & 0.20B \\ 0.55B & 0.45B \end{bmatrix}, \; Q^{2,1} = \begin{bmatrix} 0.20B & 0.80B \\ 0.60B & 0.40B \\ 0.30B & 0.70B \\ 0.80B & 0.20B \\ 0.25B & 0.75B \\ 0.80B & 0.20B \\ 0.20B & 0.80B \\ 0.45B & 0.55B \end{bmatrix}.$$

Then the beliefs $Q^{i,e}$ become effective for states $s = 1, \cdots, 8$ and the representative belief Q^* can be found:

$$Q^{1,e} = \begin{bmatrix} 0.50B & 0.50B \\ 0.80B & 0.20B \\ 0.80B & 0.20B \\ 0.90B & 0.10B \\ 0.80B & 0.20B \\ 0.65B & 0.35B \\ 0.55B & 0.45B \\ 0.80B & 0.20B \end{bmatrix}, \; Q^{2,e} = \begin{bmatrix} 0.80B & 0.20B \\ 0.60B & 0.40B \\ 0.70B & 0.30B \\ 0.80B & 0.20B \\ 0.75B & 0.35B \\ 0.80B & 0.20B \\ 0.80B & 0.20B \\ 0.45B & 0.55B \end{bmatrix}, \; Q^* = \begin{bmatrix} 0.50B & 0.50B \\ 0.60B & 0.40B \\ 0.70B & 0.30B \\ 0.80B & 0.80B \\ 0.75B & 0.25B \\ 0.65B & 0.35B \\ 0.55B & 0.45B \\ 0.45B & 0.55B \end{bmatrix}.$$

Then we have equilibrium prices $\vec{p^*} < \vec{p^\Lambda}$.

$$\vec{p^*} = \begin{bmatrix} 4.2078 \\ 4.1299 \\ 4.0519 \\ 3.9740 \\ 4.0130 \\ 4.0909 \\ 4.1688 \\ 4.2468 \end{bmatrix} < \vec{p^\Lambda} = \begin{bmatrix} 4.5 \\ 4.5 \\ 4.5 \\ 4.5 \\ 4.5 \\ 4.5 \\ 4.5 \\ 4.5 \end{bmatrix}.$$

Besides Example 5 and 6, we can also find cases in which the equilibrium prices p_s^* may be higher than p_s^Λ in some states, but lower than p_s^Λ in other states. These examples all involve social states with perfect correlation within types. Since the rationality restrictions (22) guarantee that agents are sometimes pessimistic and sometimes optimistic, when all four belief matrices are present, we can get Q^* to be the most optimistic. Then RBE will have $\vec{p^*} > \vec{p^\Lambda}$. But for RBE with perfect correlation within types, only two belief matrices are present at any time, i.e., one of the four possible cases $\{Q^{1,0}, Q^{2,0}\}$, $\{Q^{1,0}, Q^{2,1}\}$, $\{Q^{1,1}, Q^{2,0}\}$, $\{Q^{1,1}, Q^{2,1}\}$ must occur. Then there is no definite relationship between $\vec{p^*}$ and $\vec{p^\Lambda}$, since agents can become all optimistic or all pessimistic with perfect correlation within types, as demonstrated in Examples 5 and 6.

Now we study the general relationship between $\vec{p^*}$ and $\vec{p^\Lambda}$ besides those exceptional cases, when four belief matrices $\{Q^{1,0}$, $Q^{1,1}$, $Q^{2,0}$, $Q^{2,1}\}$ are all present. It also holds for the cases with three belief matrices $\{Q^{1,0}$, $Q^{1,1}$, $Q^{2,0}\}$ and $\{Q^{1,0}$, $Q^{2,0}$, $Q^{2,1}\}$, so long as one type has both belief matrices present in the economy. We call these cases "social states without perfect correlation within types." The properties of RBE in such a case were illustrated in Example 4.

Proposition 6. *For social states without perfect correlation within types, the RBE prices $\vec{p^*}$ are strictly greater than the REE prices $\vec{p^\Lambda}$ if all elements of Γ are positive and $k_L \neq \phi$ or $k_H \neq \phi$.*

This proposition applies to RBE with or without endogenous uncertainty. It helps to distinguish the Harrison and Kreps model from the present model, since in the Harrison and Kreps model no definite relation between REE prices and the minimal consistent price scheme can be proved. In our framework there is such a generic relationship. One implication of this proposition is that for RBE without endogenous uncertainty (its necessary and sufficient condition was provided in (25) and (26)), the difference $\vec{p^*} - \vec{p^\Lambda}$ can still be positive under the conditions specified in Proposition 6. This phenomenon, also called the "amplification effect," has been studied by Kurz (1998) and Wu and Guo (2002). Note that the relationship between $\vec{p^*}$ and $\vec{p^\Lambda}$ is the focus of this proposition while the properties of speculative premiums $\pi_s = p_s^* - \max_i p_s^i\,,\ s = 1, \cdots S$, were studied in Proposition 4.

4 Concluding remarks

Using unmodeled heterogeneity of expectations in economics does not have much support in the modern literature on economic research. However, the continued presence of heterogeneous expectations observed in the real world remains an important phenomenon for us to understand. It is particularly necessary when we try to study the volatile behavior of speculative trading in financial markets. This paper proposes a research framework for modeling speculative trading with diverse beliefs which are rational and compatible with the data. Even with complete learning, when all agents learn about the stationary measure, their beliefs can still remain diverse and non-stationary.

The framework of rational beliefs introduced by this paper makes the assumption of heterogeneous beliefs of Harrison and Kreps (1978) more plausible. It provides a theory of belief formation in a stable environment such that a stationary measure can be learned by all agents, but there is still left a formally defendable room for disagreement among agents (see Kurz, 1994). We demonstrate that a Rational Belief Equilibrium (RBE) exists. We also demonstrate that such a framework with rational belief can help us to understand the functioning of speculative trading in the market place. In particular, we find that Keynes' insight on speculation and subjective expectations can be supported in a rigorous framework and our paper demonstrates that heterogeneous beliefs can be the major reason for speculative trading. The extent of belief heterogeneity affects the size of speculative premiums. In such a framework, investors evaluate the asset according to its resale value and not just its dividend streams, just as described by M. Keynes. Hence, it provides a framework for pricing assets when speculative trading is allowed.

This paper also studies the phenomenon of endogenous uncertainty where equilibrium prices are distinct even when exogenous dividend states remain the same. In a framework of rational beliefs, it is essentially the uncertainty about the future beliefs of other agents. The concepts of endogenous uncertainty and rational beliefs, although formally distinct, are brought together in the last section of the paper. Our framework is in line with the thinking of M. Keynes as formulated in his "beauty contest" example. We demonstrate that endogenous uncertainty may emerge with speculative trading in a Rational Belief Equilibrium (RBE). In addition, the RBE prices are shown to be generally higher than the REE prices and speculative premiums are positive when beliefs are sufficiently diverse. These results shed light on our understanding of belief, speculation and uncertainty. Although we adopt a simple framework, the basic results can be shown to be robust (see Wu and Guo, 2002) in a more general environment with finite wealth and limited short selling.

Appendix: Proofs

To prove Propositions 1, 2 and 3, we first establish the following lemmas:

Lemma A 1. *For $S = 2$,*

$$\vec{p^i} = \frac{\gamma}{(1-\gamma)(1+\gamma(a^i-b^i))} \cdot \begin{bmatrix} (1-a^i+a^i\gamma-b^i\gamma)d_1 + a^i d_2 \\ (1-b^i)d_1 + (b^i+a^i\gamma-b^i\gamma)d_2 \end{bmatrix} \tag{A.1}$$

Proof. It is easily shown by completing the following calculation:

$$\begin{aligned} \vec{p^i} &= (I-\gamma Q^i)^{-1}\gamma Q^i \vec{d} \\ &= \left[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \gamma \begin{bmatrix} 1-a^i & a^i \\ 1-b^i & b^i \end{bmatrix}\right]^{-1} \gamma \begin{bmatrix} 1-a^i & a^i \\ 1-b^i & b^i \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \\ &= \frac{\gamma}{(1-\gamma)(1+\gamma(a^i-b^i))} \cdot \begin{bmatrix} (1-a^i+a^i\gamma-b^i\gamma)d_1 + a^i d_2 \\ (1-b^i)d_1 + (b^i+a^i\gamma-b^i\gamma)d_2 \end{bmatrix} . \end{aligned}$$

□

Lemma A 2. *If $(I-\gamma Q)\cdot\vec{x} \geq 0$ is satisfied, then each element of $\vec{x}$ must be non-negative.*

Proof. Suppose some component of $\vec{x}$ is negative. Let $x_1 < 0$ and $x_1 \leq x_s$ for all s. Then we have

$$\begin{aligned} &(1-\gamma q_{11})(x_1) + (-\gamma q_{12})(x_2) + \cdots + (-\gamma q_{1S})(x_S) \\ &= (1-\gamma)(x_1) + \gamma(q_{12}(x_1-x_2) + \cdots + q_{1S}(x_1-x_S)) < 0 \ , \end{aligned} \tag{A.2}$$

which leads to a contradiction. Hence all components of $\vec{x}$ must be non-negative. □

Proof of Lemma 1. Since $I - \gamma Q$ has a dominant diagonal, it is invertible (see McKenzie, 1960).

Proof of Proposition 1. At market equilibrium, for any state s there always exists a type of investor who has the highest subjective valuation for the asset and gets hold of it:

$$p_s^* = \max_i \gamma q_s^i(\vec{p^*} + \vec{d}), \ s = 1, 2. \tag{A.3}$$

The stationary market equilibrium price p_s^* should satisfy $p_s^* = \gamma q_s^{i(s)}(\vec{p^*} + \vec{d})$ for some types of investors $i(s)$. By applying Lemma A1 it is easily shown that

$$p_2^* - p_1^* = \frac{\gamma(b^{i(2)} - a^{i(1)})}{1 - \gamma(b^{i(2)} - a^{i(1)})}(d_2 - d_1).$$

Since $d_2 > d_1$, we have $p_2^* - p_1^* + d_2 - d_1 > 0$. Suppose state 1 occurs, the difference of willingness to pay between investors of type i and j is computed from (A1):

$$\gamma q_1^i(\vec{p^*} + \vec{d}) - \gamma q_1^j(\vec{p^*} + \vec{d}) = \gamma(a^i - a^j)((p_2^* - p_1^*) + (d_2 - d_1)),$$

which is positive if $a^i > a^j$. So the investors with maximal a^i will have the highest valuation of the asset when state 1 occurs. Similarly, when state 2 occurs the investors with maximal b^i will have the highest valuation. Substituting these back into equation (A1) and applying Lemma 1, we can derive equations (5) and (6).

Proof of Proposition 2. From Proposition 1

$$Q^* = \begin{bmatrix} 1 - \max_i a^i & \max_i a^i \\ 1 - \max_i b^i & \max_i b^i \end{bmatrix} = \begin{bmatrix} 1 - a^1 & a^1 \\ 1 - b^2 & b^2 \end{bmatrix}, \tag{A.4}$$

then by Lemma A1 we have the formula for equilibrium prices

$$\vec{p^*} = \frac{\gamma}{(1-\gamma)(1+\gamma(a^1 - b^2))} \cdot \begin{bmatrix} (1 - a^1 + a^1\gamma - b^2\gamma)d_1 + a^1 d_2 \\ (1 - b^2)d_1 + (b^2 + a^1\gamma - b^2\gamma)d_2 \end{bmatrix}. \tag{A.5}$$

By further calculation we can obtain equations (7) and (8) as in Proposition 2. We can obtain the results in (9) by differentiating equation (7) and (8).

Proof of Proposition 3. The subject valuation of a fictitious investor is

$$\vec{p^f} = (I - \gamma Q^f)^{-1} \gamma Q^f \vec{d} = \vec{p^f}(Q^f), \ f = 1, \cdots, K^S.$$

Given $\vec{p^f}$, we can find the representative investor $i(s, \vec{p^f}(Q^f))$ who has the highest valuation at state s. This defines the following mapping from Ψ into itself:

$$F(Q^f) = \begin{bmatrix} q_1^{i(1,\vec{p^f}(Q^f))} \\ \cdot \\ q_s^{i(s,\vec{p^f}(Q^f))} \\ \cdot \\ q_S^{i(S,\vec{p^f}(Q^f))} \end{bmatrix} .$$

A fixed point of this mapping implies that the associated prices also have a fixed point $\vec{p^f} = \vec{p^*}$, which is the equilibrium or minimal consistent price scheme. There are only finite elements in Ψ. If there is no fixed point of F, then there must exist a cycle $Q^{f1} \cdots Q^{fM}$ such that $F(Q^{fm}) = F(Q^{f(m+1)})$ for $m \leq M - 1$ and $F(Q^{fM}) = Q^{f1}$. The corresponding prices are $\vec{p^{f1}} \cdots \vec{p^{fM}}$. We will show in the following that $\vec{p^{fm}} = \vec{p^*},\ m = 1, \cdots, M$.

Since $F(Q^{f1}) = Q^{f2}$, as the price is equal to $\vec{p^{f1}} = \vec{p^f}(Q^f)$, the belief $q_s^{f2} = q_s^{i(s,\vec{p^{f1}}(Q^f))}$ gives the highest valuation $\gamma q_s^{f2}(\vec{p^{f1}} + \vec{d})$ of the asset and

$$\gamma q_s^{f2}(\vec{p^{f1}} + \vec{d}) \geq \gamma q_s^{f1}(\vec{p^{f1}} + \vec{d}) = p_s^{f1} \text{ for } s = 1, \cdots, S .$$

Hence we have

$$\gamma Q^{f2}(\vec{p^{f1}} + \vec{d}) \geq \vec{p^{f1}} ,$$
$$\text{or, } (I - \gamma Q^{f2})\vec{p^{f1}} \leq \gamma Q^{f2}(\vec{d}) .$$

In addition,

$$(I - \gamma Q^{f2})\vec{p^{f2}} = \gamma Q^{f2}(\vec{d}) .$$

By combining them we have

$$(I - \gamma Q^{f2})(\vec{p^{f2}} - \vec{p^{f1}}) \geq 0 .$$

Then by applying Lemma A2, all components of $\vec{p^{f2}} - \vec{p^{f1}}$ must be non-negative. Therefore, $\vec{p^{f2}} \geq \vec{p^{f1}}$ and $\vec{p^{fm}}$ increases with m. Since $\vec{p^{fm}}$ forms a cycle, all these prices must be equal to a constant $\vec{p^f}$. Then fictitious beliefs $\{Q^{fm}\}$ all come with the same price $\vec{p^f} = \vec{p^*}$, which is the equilibrium or minimal consistent price scheme.

The proof above helps us to understand the working of the algorithm mentioned in text. It can also be shown by applying Proposition 1 of Harrison and Kreps (1978) to get

$$p_s^* = \max_i \gamma q_s^i(\vec{p^*} + \vec{d}) = \gamma q^{i(s)}(\vec{p^*} + \vec{d}) \ \ s = 1, \cdots, S.$$

Then by combining them we obtain that

$$\vec{p^*} = \gamma Q^*(\vec{p^*} + \vec{d}) ,$$

where

$$q_s^* = q_s^{i(s)} \ , \ s = 1, \cdots, S.$$

Certainly Q^* is one of the members in Ψ.

Next we prove the uniqueness of equilibrium prices by contradiction. Suppose that there exist two equilibrium prices $\vec{p^{*1}}$ and $\vec{p^{*2}}$. Q^{*1} and Q^{*2} are the representative beliefs, respectively. Combining two equations $\vec{p^{*2}} = \gamma Q^{*2}(\vec{p^{*2}} + \vec{d})$ and $\vec{p^{*1}} \geq \gamma Q^{*2}(\vec{p^{*1}} + \vec{d})$, we find that

$$(I - \gamma Q^{*2})(\vec{p^{*2}} - \vec{p^{*1}}) \geq 0\,. \tag{A.6}$$

By applying Lemma A2 it follows that all components of $\vec{p^{*2}} - \vec{p^{*1}}$ must be non-negative. It can also be shown that all components of $\vec{p^{*1}} - \vec{p^{*2}}$ must be non-negative by the same argument. Then $\vec{p^{*1}} = \vec{p^{*2}}$, and so the uniqueness is proved. □

Proof of Proposition 4. We first prove the first part by two steps.
(a). No investor holds the asset in all states. If this were the case, $\vec{p^*} = \vec{p^i}$ for some i. However, since there is no dominant investor, $p^i_s < p^j_s$ or $p^i_s < q^j_s(\vec{p^* + d})$ for some s and j, a contradiction.
(b). $\vec{p^*} > \vec{p^i}$
By (a), for any i we may find $p^*_s > p^i_s$ for some s. It follows that for any given state t

$$p^*_t - p^i_t \geq \gamma q^i_t(\vec{p^*} + \vec{d}) - \gamma q^i_t(\vec{p^i} + \vec{d}) = \gamma q^i_t(\vec{p^*} - \vec{p^i}) > 0\,,$$

where the last inequality holds since $q_{tt'} > 0, \forall\, t, t'$, and $p^*_s - p^i_s$ must be positive for some s. So there exist strictly positive premiums.

Next prove the second part. Suppose that state s occurs,

$$p^*_s - p^i_s = \gamma q^*_s(\vec{p^*} + \vec{d}) - \gamma q^i_s(\vec{p^i} + \vec{d}) = m^i_s + \gamma q^i_s(\vec{p^*} - \vec{p^i}), \text{ for } s = 1, \cdots, S\,.$$

It follows that

$$\vec{p^*} - \vec{p^i} = \vec{m^i} + \gamma Q^i(\vec{p^*} - \vec{p^i})\,,$$

and then the second part of Proposition 4 is proved.

Proof of Lemma 2. Without loss of generality, let states $s = 1, \cdots \hat{s}$ correspond to the case of low dividend ($d_s = d_L$) and states $s = \hat{s}+1, \cdots S$, high dividend ($d_s = d_H$). The subjective price $\vec{p}^{\,Q}$ of an investor with belief Q will satisfy

$$\begin{bmatrix} p_1 \\ \vdots \\ p_{\hat{s}} \\ p_{\hat{s}+1} \\ \vdots \\ p_S \end{bmatrix} = \gamma \begin{bmatrix} q^Q_{1,1} & \cdots & q^Q_{1,\hat{s}} & q^Q_{1,\hat{s}+1} & \cdots & q^Q_{1,S} \\ & & & \vdots & & \\ q^Q_{\hat{s},1} & \cdots & q^Q_{\hat{s},\hat{s}} & q^Q_{\hat{s},\hat{s}+1} & \cdots & q^Q_{\hat{s},S} \\ q^Q_{\hat{s}+1,1} & \cdots & q^Q_{\hat{s}+1,\hat{s}} & q^Q_{\hat{s}+1,\hat{s}+1} & \cdots & q^Q_{\hat{s}+1,S} \\ & & & \vdots & & \\ q^Q_{S,1} & \cdots & q^Q_{S,\hat{s}} & q^Q_{S,\hat{s}+1} & \cdots & q^Q_{S,S} \end{bmatrix} \left(\begin{bmatrix} p_1 \\ \vdots \\ p_{\hat{s}} \\ p_{\hat{s}+1} \\ \vdots \\ p_S \end{bmatrix} + \begin{bmatrix} d_L \\ \vdots \\ d_L \\ d_H \\ \vdots \\ d_H \end{bmatrix} \right) \tag{A.7}$$

The necessary part is proved first. Suppose $p^Q_s = p^Q_L$ for $s = 1, \cdots \hat{s}$ and $p^Q_s = p^Q_H$ for $s = \hat{s}+1, \cdots S$. Then equation (A.7) is equivalent to

$$\begin{bmatrix} p^Q_L \\ p^Q_H \end{bmatrix} = \gamma \begin{bmatrix} \sum_{j=1}^{\hat{s}} q^Q_{s_1,j} & \sum_{j=\hat{s}+1}^{S} q^Q_{s_1,j} \\ \sum_{j=1}^{\hat{s}} q^Q_{s_2,j} & \sum_{j=\hat{s}+1}^{S} q^Q_{s_2,j} \end{bmatrix} \begin{bmatrix} p^Q_L + d_L \\ p^Q_H + d_H \end{bmatrix}, \tag{A.8}$$

for any pair s_1 and s_2 where $s_1 = 1, \cdots \hat{s}$ and $s_2 = \hat{s}+1, \cdots S$. From our discussion on 2×2 Markovian case, the subjective price in (A.8) has a unique solution. Hence we have the conditions in Lemma 2:

$$\begin{aligned} \textstyle\sum_{j=1}^{\hat{s}} q^Q_{s,j} &= k_L \text{ , for all } s = 1, \cdots \hat{s} \text{ ,} \\ \text{and } \textstyle\sum_{j=\hat{s}+1}^{S} q^Q_{s,j} &= k_H \text{ , for all } s = \hat{s}+1, \cdots S \text{ .} \end{aligned} \tag{A.9}$$

Next we prove the sufficiency part. Given that equation (17) is satisfied, then (A.7) can be reduced to

$$\begin{bmatrix} p^Q_L \\ p^Q_H \end{bmatrix} = \gamma \begin{bmatrix} k_L & 1-k_L \\ 1-k_H & k_H \end{bmatrix} \begin{bmatrix} p^Q_L + d_L \\ p^Q_H + d_H \end{bmatrix} ,$$

whose unique solution is also the solution to equation (A.8). It follows that $p^Q_s = p^Q_L$ for $s_1 = 1, \cdots \hat{s}$ and $p^Q_s = p^Q_H$ for $s_2 = \hat{s}+1, \cdots S$.

Proof of Lemma 3. In equilibrium

$$p^*_s \geq \gamma q^{i,j}_s (\vec{p^*} + \vec{d}) \text{ , for } s = 1, \cdots S \text{ and } i = 1, 2,\ j = 0, 1 \text{ ,} \tag{A.10}$$

which follows from optimizing behavior. From the rationality constraint (22) we have

$$p^*_s \geq \gamma \mathit{\Gamma}_s (\vec{p^*} + \vec{d}) \text{ , and } \vec{p^*} \geq \gamma \mathit{\Gamma} (\vec{p^*} + \vec{d}) \text{ .}$$

Since Q^{ij} has no-negative elements, it follows that

$$\gamma Q^{i,j} (\vec{p^*} + \vec{d}) \geq \gamma Q^{i,j} (\gamma \mathit{\Gamma} (\vec{p^*} + \vec{d}) + \vec{d}) \text{ .}$$

Proof of Proposition 5. (a). In general, four types of beliefs $\{Q^{1,1}, Q^{1,0}, Q^{2,1}, Q^{2,0}\}$ are all present. Proposition 3 holds for any given set of beliefs, so by Lemma 3 there exists a unique equilibrium price $\vec{p^*}$ with a fictitious representative belief Q^*.

(b). Proposition 3 provides a characterization of the equilibrium price $\vec{p^*}$ in terms of some representative belief Q^*. Then we apply Lemma 2 to such a Q^* to help to prove Proposition 5(b): Now we prove that endogenous uncertainty is generic for RBE. For any $\vec{i} = (i_s)^S_{s=1}, i_s \in \{1, \cdots, 2I\}$, we define a set $\mathcal{B}(\vec{i})$, as a subset of $\mathcal{Q} \subset \mathit{\Delta}^{2SI}$ with $\vec{Q} \in \mathcal{Q}$ as its elements satisfying

$$\begin{aligned} \sum_{k=1}^{\hat{s}} q^{i_s}_{sk} &= \sum_{k=1}^{\hat{s}} q^{i_{s'}}_{s'k}, \text{ for any } s, s'\text{=1 to } \hat{s} \text{ ,} \\ \sum_{k=\hat{s}+1}^{S} q^{i_s}_{sk} &= \sum_{k=\hat{s}+1}^{S} q^{i_{s'}}_{s'k}, \text{ for any } s, s' = \hat{s}+1 \text{ to } S \text{ ,} \end{aligned}$$

where $s = 1, \cdots \hat{s}$ correspond to the case of low dividend, $s = \hat{s}+1, \cdots S$ correspond to the case of high dividend, $\hat{s} = \frac{S}{2}$. For any $\vec{i}$, the set $\mathcal{B}(\vec{i})$ is the subset of $\mathcal{Q}$ with elements being consistent with the $2(S-1)$ linear constraints listed above.

Since it has a lower dimension, $\mathcal{B}(\vec{i})$ becomes a set with zero measure on $\mathcal{Q}$, in terms of the uniform measure $m_{\mathcal{Q}}()$:

$$m_{\mathcal{Q}}(\mathcal{B}(\vec{i})) = 0 \ .$$

We collect all possible $\mathcal{B}(\vec{i})$ to define a set $\mathcal{B}$, i.e.,

$$\mathcal{B} = \cup_{\vec{i}} \, \mathcal{B}(\vec{i}) \ .$$

Since there are only finite possible $\vec{i}$, a finite union of measure-zero sets also has a zero measure. Therefore, $\mathcal{B}$ is a set of zero measure, i.e.,

$$m_{\mathcal{Q}}(\mathcal{B}) = 0 \ .$$

Next we show that for any $\vec{Q} \in \mathcal{Q}$, if there is no endogenous uncertainty in RBE, then $\vec{Q} \in \mathcal{B}$. By the proof in Proposition 3 we can find a fixed point (for the mapping F) $Q^{f*} \in \Psi$, which is a representative belief Q^*. We define $\vec{i^*}$ as the collection of i_s^* such that

$$q_s^{i_s^*} = q_s^{f*} = q_s^* \ .$$

Since Q^* satisfies (25) and (26), such a $\vec{Q}$ should be included in $B(\vec{i^*})$. Then we can conclude that such a set of exogenous beliefs which does not lead to endogenous uncertainty in RBE has a zero measure in $\mathcal{Q}$.

(c). When there is no endogenous uncertainty,

$$\begin{bmatrix} p_L^* \\ p_H^* \end{bmatrix} = \gamma \begin{bmatrix} k_L & 1-k_L \\ 1-k_H & k_H \end{bmatrix} \begin{bmatrix} p_L^* + d_L \\ p_H^* + d_H \end{bmatrix} ,$$

which may not have the same solution as $\vec{p^\Lambda}$, which solves the following equation

$$\begin{bmatrix} p_L^* \\ p_H^* \end{bmatrix} = \gamma \begin{bmatrix} \phi & 1-\phi \\ 1-\phi & \phi \end{bmatrix} \begin{bmatrix} p_L^* + d_L \\ p_H^* + d_H \end{bmatrix} ,$$

unless $k_L = \phi$, $k_H = \phi$.

Proof of Proposition 6. For a given type i where both belief matrices are present in the economy,

$$\gamma Q^{i,j}(\vec{p^*} + \vec{d}) \leq \vec{p^*}, \text{ for } i = 0, 1 \ . \tag{A.11}$$

The above inequality and the rationality constraints of equation (22) imply that

$$\gamma \Gamma (\vec{p^*} + \vec{d}) \leq \vec{p^*} \ .$$

Combining it with $\gamma \Gamma (\vec{p^\Lambda} + \vec{d}) = \vec{p^\Lambda}$ we have

$$\vec{p^*} - \vec{p^\Lambda} \geq \gamma \Gamma (\vec{p^*} + \vec{d}) - \gamma \Gamma (\vec{p^\Lambda} + \vec{d}) = \gamma \Gamma (\vec{p^*} - \vec{p^\Lambda}) \ , \tag{A.12}$$

and then $(I - \gamma \Gamma)(\vec{p^*} + \vec{p^\Lambda}) \geq 0$. Then by Lemma A2 it leads to $\vec{p^*} - \vec{p^\Lambda} \geq 0$. Since $k_L \neq \phi$ or $k_H \neq \phi$ holds, $p_s^* - p_s^\Lambda$ must be positive for some state s by Proposition 5(c). Then by the condition that all elements of Γ are positive and (A.12), it follows that for any state s,

$$p_s^* - p_s^\Lambda \geq \gamma \Gamma_s (\vec{p^*} - \vec{p^\Lambda}) > 0 \ .$$

So $\vec{p^*}$ is strictly higher than $\vec{p^\Lambda}$.

References

Arrow, K. J.: Le Rôle des valeurs boursières pour la repartition la Meilleure des risques. Économetrie C.N.R.S. **40**, 41-47 (1953). Translated as: The role of securities in the optimal allocation of risk bearing. Review of Economic Studies **31**, 91–96 (1964)

Cass, D., Chichilnisky, G., Wu, H. M.: Individual risk and mutual insurance. Econometrica **64**, 331–341 (1996)

DeLong, J. B., Shiller, A., Summers, L. H., Waldman, R. J.: Noise trader risk in financial markets. Journal of Political Economy **98**, 703–738 (1990)

Detemple J., Murthy, S.: Intertemporal asset pricing with heterogeneous beliefs. Journal of Economic Theory **62**, 294–320 (1994)

Feiger, G.: What is speculation. Quarterly Journal of Economics **90**, 677–687 (1976)

Grossman S. J., Stiglitz, J. E.: On the impossibility of informationally efficient markets. American Economic Review **70**, 393–408 (1980)

Harris, M., Raviv, A.: Differences of opinion make a horse race. Review of Financial Studies **6**, 473–506 (1993)

Harrison, M., Kreps, D. M.: Speculative investor behavior in a stock market with heterogeneous expectation. Quarterly Journal of Economics **92**, 323–336 (1978)

Hart, O. D., Kreps, D.: Price destabilizing speculation. Journal of Political Economy **94**, 927–952 (1986)

Hirshleifer, J.: Foundations of the theory of speculation: information, risk and markets. Quarterly Journal of Economics **89**, 519–542 (1975)

Huang, P., Wu, H. M.: Market equilibrium with endogenous price uncertainty and options. In: Chichilnisky, G. (ed.) Markets, information and uncertainty: essays in honor of Kenneth Arrow, Chapter 11. New York: Cambridge University Press 1999

Kaldor, N.: Speculation and economic stability. Review of Economic Studies **7**, 1–27 (1939)

Keynes, J. M.: The general theory of employment, interest and money. London: Macmillan 1936

Kreps, D. M.: A note on "fulfilled expectations" equilibria. Journal of Economic Theory **14**, 32–43 (1977)

Kurz, M.: On the structure and diversity of rational beliefs. Economic Theory **4**, 877–900 (1994)

Kurz, M.: Social states of belief and the determinants of the equity risk premium in a rational beliefs equilibrium. In: Abramovich, Y., Avgerinos, E., Yannelis, N. C. (eds.) Functional analysis and economic theory, pp. 171–220. Berlin Heidelberg New York: Springer 1998

Kurz, M., Beltratti, A.: The equity premium is no puzzle. In: Kurz, M. (eds.) Endogenous economic fluctuations: studies in the theory of rational belief. Berlin Heidelberg New York: Springer 1997

Kurz, M., Schneider, M.: Coordination and correlation in Markov rational belief equilibria. Economic Theory **8**, 489–520 (1996)

Kurz, M., Wu, H. M.: Endogenous uncertainty in a general equilibrium model with price contingent contracts. Economic Theory **8**, 461–488 (1996)

Leach, J.: Rational speculation. Journal of Political Economy **99**, 131–144 (1991)

McKenzie, L.: Matrices with dominant diagonals and economic theory. In: Arrow, K., Karlin, S., Suppes, P. (eds.) Mathematical methods in the social sciences. Stanford: Stanford University Press 1960

Milgrom, P., Stokey, N.: Information, trade and common knowledge. Journal of Economic Theory **26**, 17–27 (1982)

Morris, S.: Speculative investor behavior and learning. Quarterly Journal of Economics **111**, 1113–1133 (1996)

Nielsen, C. K.: Rational belief structures and rational belief equilibria. Economic Theory **8**, 399–422 (1996)

Radner, R.: Existence of equilibrium plans, prices and price expectations in a sequence of markets. Econometrica **40**, 289–303 (1972)

Tirole, J.: On the possibility of speculation under rational expectation. Econometrica **50**, 1163–1181 (1982)

Varian, H. R.: Divergence of opinion in complete markets: A note. Journal of Finance **90**, 309–317 (1985)

Varian, H. R.: Differences of opinion in financial markets. In: Stone, C. C. (eds.) Financial risk: theory, evidence and implications-proceedings of the Eleventh Annual Economic Policy Conference of the Federal Reserve Bank of St. Louis 1989

Wu, H. M., Guo, W. C.: Asset market speculation with a large number of rational agents. Working Paper (2002)