Two Mathematical Versions of the Coase Theorem

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Summary: This paper models the major aspects and main insights of the Coase theorem by two mathematical theorems: In coalitional TU (NTU) economies without transaction costs, maximal (efficient) payoffs are produced by allocating inputs to optimal (efficient) firms based on firms' shadow values and will be split within the non-empty core. These results follow from the equivalence between firm's shadow value and balancing weight. They provide not only a fix for the empty-core problem of the Coase theorem (Aivazian and Callen 1981, Coase 1981) but also a method for estimating the bounds of transaction costs in each application of the Coase theorem.

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1. Introduction

This paper uses the model of coalitional economies (also called bargaining economies or coalitional games) with transferable utilities (TU) and with non-transferable utilities (NTU) to study the Coase theorem,[1] and it shows that the main insights and major aspects of the Coase theorem can be stated in two mathematical theorems: In coalitional TU (NTU) economies without transaction costs, maximal (efficient) payoffs are produced by allocating inputs to optimal (efficient) firms based on firms' shadow values, and these payoffs will be split by the owners within the non-empty core.

The paper also makes three other advances: 1) It fixes the empty-core problem of the Coase theorem (Aivazian and Callen 1981, Coase 1981), which appeared so serious that Telser (1994) claimed that "the Coase 'theorem' needs much repair when there is an empty core"; 2) it provides a method to estimate the upper (lower) bound of transaction costs that support (prevent) the optimal outcome predicted by the Coase theorem in each application; and 3) it establishes the duality between sub-coalitions' producing ability (which defines the maximal payoff) and blocking power (which defines the core stability). Such duality reveals that a firm's shadow value is equal to its balancing weight, and it implies six theorems (three new and three known) on the existence of the usual core in coalitional games.

The breakthrough in our search for a repair of the empty-core problem is the discovery that maximal profits in the Aivazian-Callen example (1981), or efficient outcomes in more general cases, are achieved in advanced forms of production or minimal balanced collections of the firms rather than conventional forms of production such as the monopoly

[1] The earliest reference to (or version of) Coase theorem was in Stigler (1966, Chapter 6, page 113): "The Coase theorem thus asserts that under perfect competition private and social costs will be equal".

or other partitions of the firms.

The rest of the paper is organized as follows. Section 2 provides a fix for the empty-core problem in the Aivazian-Callen example (1981). Section 3 studies the duality between sub-coalitions' blocking power and producing ability in coalitional TU games. Sections 4 and 5 establish the TU and NTU Coase theorems, respectively. Section 6 concludes, and the appendix provides proofs.

2. A fix for the empty-core example of the Coase theorem

An important study of the Coase theorem was the following empty-core example reported more than three decades ago in Aivazian and Callen (1981):

Example 1: *$n=3$, $v(1)=3000$, $v(2)=8000$, $v(3)=24000$; $v(12)=15000$, $v(13)=31000$, $v(23)=36000$; $v(123)=40000$, where for each firm $S \subseteq N=\{1,2,3\}$, $v(S)$ is its daily profits.*[2]

This is an example of a three-owner coalitional economy or three-person coalitional TU game. Its five partitions or conventional forms of production are: $\mathcal{B}_1=\{1,2,3\}$, $\mathcal{B}_2=\{12,3\}$, $\mathcal{B}_3=\{13,2\}$, $\mathcal{B}_4=\{23,1\}$, and $\mathcal{B}_m=\{123\}$, their total profits satisfy $\pi(\mathcal{B}_1)=v(1)+v(2)+v(3)=35000 < \pi(\mathcal{B}_2)=v(12)+v(3)=\pi(\mathcal{B}_3)=\pi(\mathcal{B}_4)=39000<\pi(\mathcal{B}_m)=v(123)=40000$. Given that the monopoly has the largest profits among the five partitions, the Coase theorem implies that the monopoly merger will be formed. However, this conclusion breaks down after one checks coalitional rationality. Let $x=(x_1,x_2,x_3)\geq 0$ be a split of $\pi(\mathcal{B}_m)=40000$. Subcoalitions' rationality requires that it has no blocking coalitions or be in the usual core or satisfy the following inequalities:

(i) $x_1 \geq v(1)=3000$, $x_2 \geq v(2)=8000$, $x_3 \geq v(3)=24000$;

(ii) $x_1+x_2 \geq v(12)=15000$, $x_1+x_3 \geq v(13)=31000$, and $x_2+x_3 \geq v(23)=36000$.

[2] We simplify $v(\{i\})$ as $v(i)$, $v(\{1,2\})$ as $v(12)$. Similar simplifications apply to other coalitions.

Adding up the inequalities in (*ii*) yields $x_1+x_2+x_3 \geq (15000+31000+36000)/2 = 41000 > 40000$, which contradicts $x_1+x_2+x_3 = 40000$. Thus any split of monopoly profits is blocked by at least one coalition. Consequently, the monopoly can not be formed so the Coase theorem fails.

Is the above argument false, or could the insight of the Coase theorem possibly be wrong? Telser (1994) observed that "Coase's elaborate analysis in his comment (1981) fails to come to grip with the issues raised by this example," and concluded that "The Coase "theorem" needs much repair when there is an empty core" (see Aivazian and Callen 2003 for discussion). Such a serious issue remained unsettled for more than three decades, until this study. To see our repair, move a step deeper inside our coalitional economy and assume that each of the seven firms produces a product called profit from an input called labor, using the following linear production functions:

$f_1(x) = 3000x/8 = 375x$, $f_2(x) = 1000x$, $f_3(x) = 3000x$, $0 \leq x \leq 8$;

$f_{12}(x) = 937.5x$, $f_{13}(x) = 1937.5x$, $f_{23}(x) = 2250x$, $0 \leq x \leq 16$; and $f_{123}(x) = 5000x/3$, $0 \leq x \leq 24$;

where a singleton firm has one full-time worker or 8 hours of labor inputs, so two-party firms have two workers, and the monopoly, three workers. It is straightforward to see that these production functions generate the same profits as those in Example 1.

Now, consider operating each of the three two-party firms at full capacity for 4 hours (or at half capacity for 8 hours), which can be arranged, for example, in the following sequence: $S=12$ opens at full capacity from 8:00 *a.m.*-noon, $S=13$ from noon-4:00 *p.m.*, and $S=23$ from 4:00-8:00 *p.m.* The profits from this advanced form of production are truly maximal and are given by

$$mp = f_{12}(8) + f_{13}(8) + f_{23}(8) = [v(12) + v(13) + v(23)]/2 = 41000 > \pi(\mathcal{B}_m) = 40000,$$

which exceeds the profits of operating the monopoly at full capacity for 8 hours. Define the new core as the splits of the above *mp* that are unblocked by all subcoalitions. One can check that the new core of Example 1 has a unique vector: x_1=5000, x_2=10000, x_3=26000, which is the optimal outcome predicted by the Coase theorem.

Note that the above optimal outcome is achieved when all three firms open for business only half of the time and all three workers have two part-time jobs, instead of one full-time job. This conclusion does not seem to have been reported in the previous literature, and it provides a new argument in studying labor theory and production theory.

Now, our repair is to replace conventional forms of production with advanced forms of production, or replace the usual core with the always non-empty new core (or simply, the core), so the Coase theorem is now free of the empty-core problem and will be precisely stated as Theorem 2 in section 4 and Theorem 4 in section 5.

3. The maximum of generated-payoffs and the duality in coalitional TU games

Let $N=\{1, 2, ..., n\}$ be the set of players, $\mathcal{N}=2^N$ be the set of all coalitions. A TU game in coalitional (or characteristic) form is given by a set function $v\colon \mathcal{N}\rightarrow \boldsymbol{R}_+$ with $v(\emptyset)=0$, specifying a joint payoff $v(S)$ for each coalition $S\in\mathcal{N}$, or precisely by

(1) $\Gamma = \{N, v(\cdot)\}$.

We will refer game *(1)* as a coalitional economy and a player i as the owner of firm i (or worker i) when the emphasis is on the Coase theorem. We use a lowercase v in $v(\cdot)$ to define the above TU game *(1)*, and an uppercase V in $V(\cdot)$ to define NTU games in section 5.

Let $X(v(N)) = \{x\in \boldsymbol{R}^n_+ \,|\, \Sigma_{i\in N}x_i = v(N)\}$ denote the *preimputation space* or the set of payoff vectors that are splits of $v(N)$. A split $x=(x_1,...,x_n)\in X(v(N))$ satisfies the rationality of a coalition $S\in\mathcal{N}$ or is unblocked by S if it gives S no less than $v(S)$ (i.e., $\Sigma_{i\in S}x_i \geq v(S)$), and is in the usual core if it is unblocked by all $S\neq N$. Denote the usual core of *(1)* as

(2) $c_0(\Gamma) = \{x \in X(v(N)) \mid \Sigma_{i\in S} x_i \geq v(S) \text{ for all } S \neq N\}$.

We use a lowercase *c* in $c_0(\Gamma)$ to denote the above TU core and an uppercase *C* in $C_0(\Gamma)$ to denote the NTU core in section 5.

The concept of generated-payoffs is defined by balanced collections of coalitions. Given a player $i \in N$ and a collection of coalitions $\mathcal{B} = \{T_1, \ldots, T_k\}$, let $\mathcal{B}(i) = \{T \in \mathcal{B} \mid i \in T\}$ denote the subset of coalitions of which *i* is a member. Then, $\mathcal{B}$ is a balanced collection if it has a balancing vector $w = \{w_T | T \in \mathcal{B}\} \in \boldsymbol{R}^k_{++}$ such that $\Sigma_{T\in\mathcal{B}(i)} w_T = 1$ for each $i \in N$.

To see the intuition of a balancing vector, treat game *(1)* as a coalitional economy as in Example 1, where each singleton firm *i* has one full-time worker (or *8* hours of labor inputs). For each firm $S \in \mathcal{N}$, let $k = k(S) = |S|$ denote its size or cardinality, then each firm *S* has *8k* hours of labor inputs or *k* workers, and produces a daily profit of *v(S)* by operating at full capacity for *8* hours, with the following linear production function:

(3) $f_S(x) = v(S)x/[8k(S)], \ 0 \leq x \leq 8k(S)$.

Now, each balanced $\mathcal{B}$ with a balancing vector *w* defines the following form of production:

Operate each firm $T \in \mathcal{B}$ at full capacity for $8w_T$ hours, and produce a profit equal to

(4) $gp(\mathcal{B}) = \Sigma_{T\in\mathcal{B}} f_T(8 w_T k(T)) = \Sigma_{T\in\mathcal{B}} w_T v(T)$.

The above operation and its generated payoff $gp(\mathcal{B})$ are feasible because the condition $\Sigma_{T\in\mathcal{B}(i)} w_T = 1$ or balancedness ensures that each worker *i* works exactly *8* hours (i.e., *i* works for $8w_T$ hours at each *T* in $\mathcal{B}(i)$). In Example 1, the collection $\mathcal{B}_5 = \{12, 13, 23\}$ with balancing vector $w = \{½, ½, ½\}$ yields the payoff $gp(\mathcal{B}_5) = [v(12) + v(13) + v(23)]/2 = 41000$.

Thus, balancing weights here are proportions of inputs that each firm $T \in \mathcal{B}$ receives from all its owners (or that a worker *i* allocates to his/her firms in $\mathcal{B}(i)$). Such productions represent an advance beyond conventional forms of production or partitions, because they include partitions as special cases and they could, as in Example 1, possibly produce higher payoffs. The discovery of such higher payoffs beyond $v(N)$ is the breakthrough in our search for a repair for the empty-core problem of the Coase theorem.[3]

A balanced collection is minimal if no proper subcollection is balanced. It is known that a balanced collection is minimal if and only if its balancing vector is unique (Shapley 1967). Denote the set of all minimal balanced collections (excluding the grand coalition) as

(5) $$B = \{\mathcal{B} = \{T_1, ..., T_k\} \mid N \notin \mathcal{B},\ \mathcal{B} \text{ is a minimal balanced collection}\}.$$

In three-person games like Example 1, B has five entries: the four non-monopoly partitions, $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$ and $\mathcal{B}_4$, plus $\mathcal{B}_5 = \{12, 13, 23\}$. We are now ready to define the maximum of generated-payoffs (*mgp*) and maximal payoffs (*mp*):[4]

Definition 1: *Given game (1), gp(*$\mathcal{B}$*) in (4), and B in (5), mgp and mp are given by*

(6) $$mgp = mgp(\Gamma) = Max\ \{gp(\mathcal{B}) \mid \mathcal{B} \in B\}, \text{ and}$$

(7) $$mp = mp(\Gamma) = Max\ \{mgp, v(N)\}.$$

The definition considers only minimal balanced collections because *mgp* is achieved among minimal balanced collections, just as the optimal value in linear programming is achieved among the extreme points. The following duality result is the theoretical foundation of our repair for the empty-core problem of the Coase theorem.

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[3] It is known that total payoffs in a non-monopoly partition might be higher than the monopoly payoffs. See Sun et al (2008) on market games, Zhou (1994) on bargaining set, and Guesnerie and Oddou (1979) on c-core.

[4] It is easy to show that *mp* is equal to grand coalition's payoff in the game's cover (Shapley-Shubik 1969). When the functions in (3) satisfy $f_S(x) = \Sigma_{i\in S} f_i(\lambda_i)$ for each S, all $\lambda \geq 0 \in R^S$, $\Sigma_{i\in S}\lambda_i = x$, game (1) becomes identical to a market (T, G, A, U) as defined in Shapley-Shubik (1969), with $T = N$, $G = R^n_+$, $A = I$, and $U = \{f_i \mid i \in N\}$.

Theorem 1: *Given game (1), the maximization problem (6) for mgp is dual to the following minimization problem for the minimum no-blocking payoff (mnbp,* Zhao 2001*):*

(8) $$mnbp = Min\ \{\Sigma_{i\in N} x_i \mid x \in \boldsymbol{R}^n_+;\ \Sigma_{i\in S}\, x_i \geq v(S),\ \text{all } S \neq N\},$$

so mgp = mnbp holds.

By the above duality, a firm's shadow value in *(8)* is equal to the balancing weight[5] in *(6)*, and the minimal worth of the grand coalition needed to guarantee no-blocking given in *(8)* is equal to the maximal payoffs produced by sub-coalitions given in *(6)*. Because *mnbp* represents sub-coalitions' power to block proposed splits of $v(N)$ (Zhao 2001), *mgp* represents their ability to produce payoffs that are different from $v(N)$, their producing ability and blocking power are dual to each other. Such duality is perhaps the most salient property in cooperative games, because it also holds in NTU games (see Theorem 3) and it leads directly to three existence theorems (two known) on the usual core, as given below:

Lemma 1: *Given game (1), its usual core is non-empty if and only if each of the following three arguments holds:*

(i) the game is balanced (Bondareva 1962, Shapley 1967);

(ii) the grand coalition has enough to guarantee no-blocking (Zhao 2001)*; and*

(iii) players can not produce a higher payoff than the grand coalition's payoff.

Precisely, the above three core arguments are: *(i)* $\Sigma_{T\in\mathcal{B}} w_T v(T) \leq v(N)$ for each balanced $\mathcal{B}$ with a balancing vector w, *(ii)* $v(N) \geq mnbp$, and *(iii)* $mgp \leq v(N)$. Theorem 1 also implies, as shown in the next section, the answers to four important questions arising from the Coase

[5] So far we have three interpretations of the balancing weights: *i*) percentage of time during which the firm operates; *ii*) proportion of inputs that the firm receives; and *iii*) the frequency (or probability) with which the firm forms (or a player joins his/her coalition), assuming that the game is replicated/repeated for a finite number of times (or that uncertainty is added into the game). Other interpretations remain to be discovered. The shadow value for the grand coalition N can be given as: $w_N = 1$ if $v(N) \geq mgp$, and $w_N = 0$ if $v(N) < mgp$.

theorem: What payoffs will be split? How will the payoff be split? What firms will form? and How much inputs will each of the formed firms receive? Note that the Coase theorem won't be complete if any of these four questions is not answered.

4. The TU Coase theorem

By the new argument for core existence or part (iii) in Lemma 1, players in games with an empty usual core will not split the grand coalition's payoff $v(N)$, because the game's *mgp* is greater than $v(N)$. Then, what payoffs will they split? We postulate that they split the maximal payoff *mp* in *(7)*. By $mp = v(N)$ if $c_0(\Gamma) \neq \varnothing$, $= mgp > v(N)$ if $c_0(\Gamma) = \varnothing$, it stands to reason that they will always split *mp*. This answers the question of what payoffs will be split.

Next, consider the question of how to split the maximal payoff. Let

(9) $$Y = Y(\Gamma) = Arg\text{-}Min\{\Sigma_{i\in N} x_i \mid x \in \boldsymbol{R}^+_n,\ \Sigma_{i\in S}\, x_i \geq v(S) \text{ for all } S \neq N\}$$

denote the minimal set or the set of minimal solutions in *(8)* for *mnbp*. Coalitional rationality or no-blocking leads to the following new core:

(10) $$c(\Gamma) = \{x \in X(mp) \mid \Sigma_{i\in S} x_i \geq v(S), \text{ all } S \subseteq N\} = \begin{cases} c_0(\Gamma) \text{ if } v(N) = mp \\ Y(\Gamma) \text{ if } v(N) < mp, \end{cases}$$

which answers the question of how to split *mp*. Note that the above new core or simply the core is always nonempty because it is identical to the usual core in *(2)* when the usual core is nonempty, and the minimal set in *(8) or (9)* when the usual core is empty.

Note that the new core in *(10)* includes the usual core of the game's cover or balanced cover (Shapley-Shubik 1969), which is a new game $\Gamma_{bc} = \{N, \bar{v}(\cdot)\}$, where each $\bar{v}(S) = mp(S)$ is the maximal payoff of the subgame $\Gamma_S = \{S, v(\cdot)\}$.

Now, consider the question of what firms will form. Because players always split *mp*, they will form the optimal collections or optimal coalitions that generate *mp*, which will be either the grand coalition or the optimal set in *(6)* for *mgp* or their union. Let

(11) $$B_0 = B_0(\Gamma) = \{\mathcal{B}\in B \mid gp(\mathcal{B}) = mgp\} = Arg\text{-}Max\{gp(\mathcal{B}) \mid \mathcal{B}\in B\}$$

denote the optimal set in *(6)*, then the set of optimal collections $B^*(\Gamma)$ can be given as

(12) $$B^* = B^*(\Gamma) = \begin{cases} \{N\} & \text{if } mgp(\Gamma) < v(N); \\ B_0(\Gamma) & \text{if } mgp(\Gamma) > v(N); \\ \{N\} \cup B_0(\Gamma) & \text{if } mgp(\Gamma) = v(N); \end{cases}$$

which answers the question of what firms will form.

Finally, the unique balancing vector for each optimal $\mathcal{B}$ answers the question of how much inputs will each of the formed firms receive. The above answers cover all the relevant aspects of the Coase theorem for our coalitional economy *(1)*, which can now be stated as:

Theorem 2: *In coalitional economy (1) without transaction costs, owners will produce the maximal payoff by allocating inputs to optimal firms based on firms' shadow values, and they will split the maximal payoff within the non-empty core.*

Precisely, the maximal payoff, core and optimal firms are respectively given in *(7)*, *(10)* and *(12)*, and the balancing vector for each set of optimal firms in *(12)* specifies their shadow values or proportions of inputs received from the owners.

Theorem 2 provides a method to estimate the bounds of transaction costs (or merging costs) for each firm or coalition *S*, using the approach introduced by the author (Zhao 2009). For simplicity, let the merging costs for monopoly be $\tau_N > 0$ and that for all sub-coalitions be zero in our game (1). Because the owners now can only split $[v(N)\text{-}\tau_N]$ and *mgp* = *mnbp* remains unchanged, monopoly formation now requires:

(13) $$[v(N)\text{-}\tau_N] \geq mnbp,$$

so $\tau_N \leq [v(N)\text{-}mnbp]$ ($\tau_N > [v(N)\text{-}mnbp]$) holds for a successful (failed) monopoly formation.

Thus, the difference between *v(N)* and *mnbp* serves as an upper (lower) bound of monopoly's transaction costs below (above) which monopoly formation is possible (impossible), which can be empirically estimated.

5. The NTU Coase theorem

This section studies the efficient payoffs and efficient firms in an NTU coalitional economy. An NTU coalitional economy (also called NTU bargaining economy, NTU coalitional game, and NTU game in characteristic form) is defined as

(14) $\Gamma = \{N, V(\cdot)\}$,

which specifies, for each $S \in \mathcal{N}$, a non-empty set of payoffs $V(S)$ in $\boldsymbol{R}^S$, the Euclidean space whose dimension and coordinates are the number of players in S and their payoffs. Let

$$\partial V(S) = \{ y \in V(S) \mid \text{there is no } x \in V(S) \text{ such that } x >> y\},$$

denote the (weakly) efficient set of each $V(S)$, where vector inequalities are defined by: $x \geq y \Leftrightarrow x_i \geq y_i$, all i; $x > y \Leftrightarrow x \geq y$ and $x \neq y$; and $x >> y \Leftrightarrow x_i > y_i$, all i.

Scarf (1967b) introduced the following two assumptions for (14): (i) each $V(S)$ is closed and comprehensive (i.e., $y \in V(S)$, $u \in \boldsymbol{R}^S$ and $u \leq y$ imply $u \in V(S)$); (ii) for each S, $\{y \in V(S) \mid y_i \geq \partial V(i) > 0$, all $i \in S\}$ is non-empty and bounded, where $\partial V(i) = Max\{x_i \mid x_i \in V(i)\}$. Under these two assumptions, each $\partial V(S)$ is closed, non-empty and bounded.

Given $S \in \mathcal{N}$, a payoff vector $u \in \boldsymbol{R}^n_+$ is blocked by S if S can obtain a higher payoff for each of its members than that given by u, or precisely if there is $y \in V(S)$ such that $y >> u_S = \{u_i \mid i \in S\}$ or $u_S \in V(S) \backslash \partial V(S)$. A payoff vector $u \in \partial V(N)$ is in the usual core if it is unblocked by all $S \neq N$, so the usual core of (14) can be given as

(15) $C_0(\Gamma) = \{ u \in \partial V(N) \mid u_S \notin V(S) \backslash \partial V(S), \text{ all } S \neq N \}$.

Balanced NTU games can be defined geometrically as below. For each $S \neq N$, let $\tilde{v}(S)=$

$V(S)\times\boldsymbol{R}^{-S}\subset\boldsymbol{R}^n$, where $\boldsymbol{R}^{-S}=\prod_{i\in N\setminus S}\boldsymbol{R}^i$. For each minimal balanced collection $\mathcal{B}\in B$ in *(5)*, let

$$(16)\qquad GP(\mathcal{B}) = \cap_{S\in\mathcal{B}}\tilde{v}(S), \text{ and } GP = \cup_{\mathcal{B}\in B} GP(\mathcal{B})$$

denote the payoffs generated by $\mathcal{B}$ and the set of generated-payoffs. Note that $GP(\mathcal{B})$ is simplified to $GP(\mathcal{B})=\prod_{S\in\mathcal{B}}V(S)$ when $\mathcal{B}$ is a partition. Similar to the TU case, we only need to consider minimal balanced collections because non-minimal balanced collections don't generate additional payoffs. Now, we are ready to define the efficient generated-payoffs (*EGP*) and efficient payoffs (*EP*), which are the NTU counterparts of *mgp* and *mp* in *(6)-(7)*.

Definition 2: *Given game (14) and its GP in (16), its EGP and EP are given by*

$$(17)\qquad EGP = \partial GP = \{y\in GP \mid \exists \text{ no } x\in GP \text{ such that } x>>y\}, \text{ and}$$

$$(18)\qquad EP = EP(\Gamma)=\partial(GP\cup V(N))= \{y\in GP\cup V(N) \mid \exists \text{ no } x\in GP\cup V(N) \text{ with } x>>y\}.$$

Readers are encouraged to visualize the generated-payoffs in the following example, which are illustrated in Figure 1.

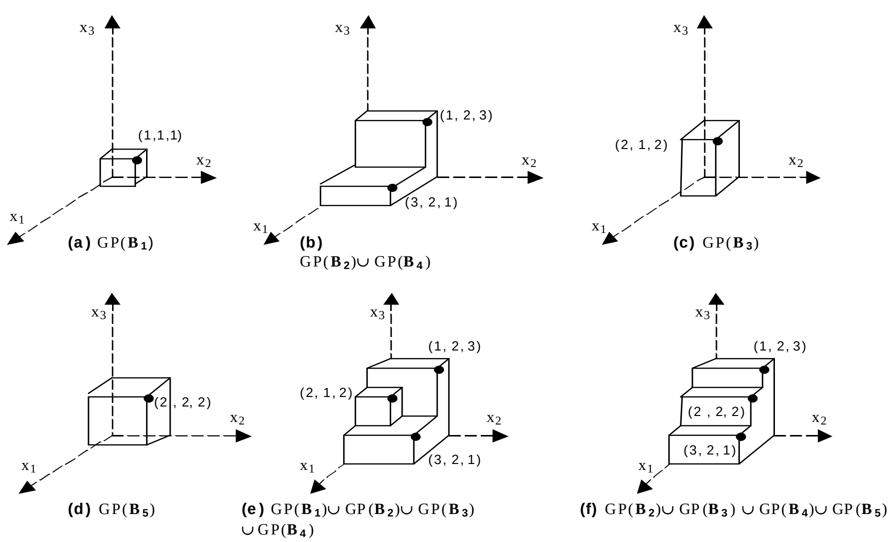


Figure 1. The generated payoffs in Example 2, where $B_1= \{1, 2, 3\}$, $B_2 = \{12, 3\}$, $B_3 = \{13, 2\}$, $B_4 = \{23, 1\}$, and $B_5 = \{12, 13, 23\}$.

Example 2: $n = 3$, $V(i) = \{x_i | x_i \leq 1\}$, $i = 1,2,3$; $V(12) = \{(x_1, x_2) | (x_1, x_2) \leq (3,2)\}$, $V(13) = \{(x_1, x_3) | (x_1, x_3) \leq (2,2)\}$, $V(23) = \{(x_2, x_3) | (x_2, x_3) \leq (2,3)\}$, $V(123) = \{x | x_1 + x_2 + x_3 \leq 5\}$. *Let* $\mathcal{B}_i$, $i = 1,2,3,4$, *be as in Example 1, and* $\mathcal{B}_5 = \{12, 13, 23\}$. *Then,* $GP(\mathcal{B}_1) = \{x | x \leq (1,1,1)\}$, $GP(\mathcal{B}_2) = \{x | x \leq (3,2,1)\}$, $GP(\mathcal{B}_3) = \{x | x \leq (2,1,2)\}$, $GP(\mathcal{B}_4) = \{x | x \leq (1,2,3)\}$, *and* $GP(\mathcal{B}_5) = \{x | x \leq (2,2,2)\}$.

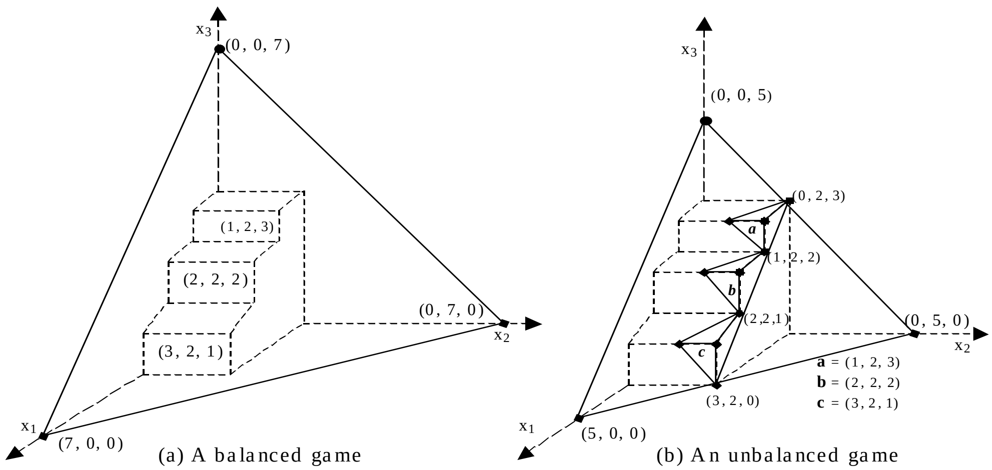


Figure 2. Balanced and unbalanced games.

Now, the game (14) is balanced if $GP(\Gamma) \subset V(N)$ or if for each balanced $\mathcal{B}$, $u \in V(N)$ must hold if $u_S \in V(S)$ for all $S \in \mathcal{B}$. To see a balanced game geometrically, visualize that one is flying over a city. Treat the generated-payoffs *GP* as trees and buildings in the city and the grand coalition's payoff *V(N)* as clouds. Then, a game is balanced if one sees only clouds and unbalanced if one sees at least one building or tree top above the clouds. In Figure 2b for Example 2, one sees three building tops above the clouds so the game is unbalanced. In Figure 2a for Example 3, one sees only clouds so the game is now balanced.

Example 3: *Same as Example 2 except* $V(123) = \{x | x_1 + x_2 + x_3 \leq 7\}$.

Note that the collection $\mathcal{B}_5 = \{12, 13, 23\}$ in Example 2 generates new payoffs that are outside of those generated by the four partitions and are better than *v(N)*, see the difference between [e] and [f] in Figure 1. Needless to say, it is the discovery of such new and better payoffs that gives rise to our mathematical versions of the Coase theorem.

Recall that a payoff vector u is unblocked by S if $u \in [V(S) \backslash \partial V(S)]^C \times \boldsymbol{R}^{-S} \subset \boldsymbol{R}^n$ or if $u_S \notin V(S) \backslash \partial V(S)$, where the superscript C denotes the complement of a set. The following concept of minimum no-blocking frontier is the NTU counterpart of *mnbp* in (8):

Definition 3: *Given game (14), the set of payoffs unblocked by all $S \neq N$ (UBP) and the minimum no-blocking frontier (MNBF) are given, respectively, as*

(19) $$UBP = UBP(\Gamma) = \bigcap_{S \neq N} \{[V(S) \backslash \partial V(S)]^C \times \boldsymbol{R}^{-S}\} \subset \boldsymbol{R}^n, \text{ and}$$

(20) $$MNBF = MNBF(\Gamma) = \partial UBP = \{y \in UBP \mid \exists \text{ no } x \in UBP \text{ such that } x << y\}.$$

It is easy to see that each payoff vector on or above *MNBF* is unblocked by all $S \neq N$, and the usual core can be given as $C_0(\Gamma) = UBP \cap \partial V(N) = MNBP \cap \partial V(N)$. Similar to the TU case, *MNBF* represents sub-coalitions' power to block the grand coalition's proposals. Theorem 3 below shows that sub-coalitions' blocking power and producing ability are also dual to each other in coalitional NTU games, which is the NTU counterpart of Theorem 1.

Theorem 3: *Given game (14), its minimum no-blocking frontier and efficient generated-payoffs have a non-empty intersection.*

To put it differently, the NTU counterpart of *mnbp* = *mgp* in game (1) is

(21) $$Z = Z(\Gamma) = MNBF \cap EGP \neq \varnothing.$$

It is straightforward to verify $a, b, c \in Z$ in Example 2 (see Figure 2b), where $a = \{1, 2, 3\}$, $b = \{2, 2, 2\}$, and $c = \{3, 2, 1\}$, so $Z \neq \varnothing$ holds in the example.

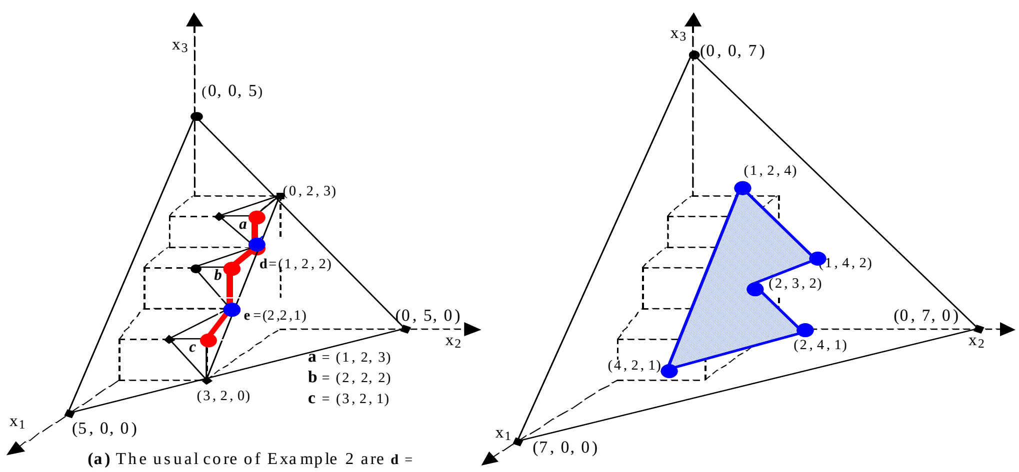
Recall that $EGP \subseteq V(N)$ holds in balanced games. Then, $MNBF \cap EGP \neq \varnothing$ implies $MNBF \cap \partial V(N) = C_0(\Gamma) \neq \varnothing$ in balanced games. Hence, the above duality implies Scarf's core theorem (1967b). The above duality also implies two new existence theorems on the usual core, which are summarized in the following lemma:

Lemma 2: *Given game (14), the following three claims hold:*

(i) its usual core is non-empty if it is balanced (Scarf 1967b);

(ii) its usual core is non-empty if and only if the grand coalition has enough to guarantee no blocking; and

(iii) its usual core is non-empty if players can't produce better payoffs than the grand coalition's payoff.



(a) The usual core of Example 2 are **d** = (1,2,2) and **e** = (2,2,1), the new core is the segment of the edge linking a, d, b, e, c.

(b) The usual core of Example 3.

Figure 3. The usual core and the new core: payoffs in the usual core are blue-colored, and payoffs in the new core are red-colored.

The above results in (i) and (iii) can be precisely stated in one argument: $C_0(\Gamma)\neq\varnothing$ if $GP\subset V(N)$, and the result in (ii) can be precisely stated as: $C_0(\Gamma)\neq\varnothing \Leftrightarrow$ there exists $x\in\partial V(N)$ and $y\in MNBF$ such that $x\geq y$.

Due to the generality of non-transferable utilities, the above NTU core results are more general than the earlier TU results in at least three aspects: *I*) the usual NTU core is no longer convex, as shown in Figure 3 for Examples 2-3; *II*) balancedness is only sufficient and no longer necessary for a non-empty usual NTU core, see a non-empty usual NTU core in an unbalanced game in Figure 3a for Example 2; and *III*) "players can't produce better payoffs than *V(N)*" is no longer necessary for a non-empty usual NTU core.

Recall that $Z(\Gamma)$ in *(21)* is the set of unblocked and efficient generated-payoffs. Let

(22) $$Z(\Gamma)^* = Z(\Gamma) \cap [V(N) \backslash \partial V(N)]^C$$

denote the subset of $Z(\Gamma)$ that are also unblocked by the grand coalition *N*. Now, let the set of minimal balanced collections that generate $Z(\Gamma)$ and $Z(\Gamma)^*$ be denoted respectively by

(23) $$D_0(\Gamma) = \{\mathcal{B} \in B \,|\, GP(\mathcal{B}) \in Z(\Gamma)\}, \text{ and}$$

(24) $$D_1(\Gamma) = \{\mathcal{B} \in D_0(\Gamma) \,|\, GP(\mathcal{B}) \in Z(\Gamma)^*\}.$$

Note that $D_0(\Gamma)$ is the NTU counterpart of the minimal set $B_0(\Gamma)=\{\mathcal{B}\in B \,|\, gp(\mathcal{B}) = mgp\}$ in *(11)*. Then, replacing the grand coalition's payoff *V(N)* with the efficient payoffs *EP* in *(18)* yields the following new NTU core $C(\Gamma)$ and set of efficient firms $D^*(\Gamma)$:

(25) $$C(\Gamma) = \{\, u \in EP(\Gamma) \,|\, u_S \notin V(S) \backslash \partial V(S), \text{ all } S \subseteq N \,\}$$

$$= \begin{cases} C_0(\Gamma) & \text{if } GP(\Gamma) \subset V(N) \\ Z(\Gamma) & \text{if } V(N) \subset GP \backslash \partial GP \text{ or if } V(N) \not\subset GP \backslash \partial GP, GP \not\subset V(N), C_0(\Gamma) = \varnothing \\ C_0(\Gamma) \cup Z(\Gamma)^* & \text{if } V(N) \not\subset GP \backslash \partial GP, GP \not\subset V(N) \text{ and } C_0(\Gamma) \neq \varnothing, \end{cases}$$

(26) $$D^*(\Gamma) = \begin{cases} \{N\} & \text{if } GP(\Gamma) \subset V(N) \\ D_0(\Gamma) & \text{if } V(N) \subset GP \backslash \partial GP \text{ or if } V(N) \not\subset GP \backslash \partial GP, GP \not\subset V(N), C_0(\Gamma) = \varnothing \\ \{N\} \cup D_1(\Gamma) & \text{if } V(N) \not\subset GP \backslash \partial GP, GP \not\subset V(N) \text{ and } C_0(\Gamma) \neq \varnothing, \end{cases}$$

where $C_0(\Gamma)$, $GP(\Gamma)$, $Z(\Gamma)$, $Z(\Gamma)^*$, $D_0(\Gamma)$ and $D_1(\Gamma)$ are, respectively, the usual core in *(15)*, generated payoffs in *(15)*, unblocked and efficient generated-payoffs in *(21)*, efficient generated-payoffs that are also unblocked by grand coalition in *(22)*, collections supporting $Z(\Gamma)$ in *(23)*, and collections supporting $Z(\Gamma)^*$ in *(24)*.

In words, the new NTU core or simply the NTU core is characterized in three cases, it is equal to: *(i)* the usual core if the game is balanced; *(ii)* the set of unblocked and efficient generated-payoffs if players can produce better payoffs than *V(N)* or if players can not produce better payoffs than *V(N)* and the game is unbalanced with an empty usual core; and *(iii)* the union of the usual core and a subset of the second case if players can not produce better payoffs than *V(N)* and the game is unbalanced with a non-empty usual core. Figure 3a illustrates the difference between the usual NTU core (i.e., points *d* and *e*) and new NTU core (i.e., the segment of the edge linking all three peaks) in Example 2.

Note that efficient firms in *(26)* are defined according to the three cases of the core in *(25)*. There are five sets of efficient firms[6] in Example 2: $\{N\}$, $\mathcal{B}_2=\{12,3\}$, $\mathcal{B}_3=\{13,2\}$, $\mathcal{B}_4=\{23,1\}$, and $\mathcal{B}_5=\{12,13,23\}$. Now, our NTU Coase theorem, comprising the above answers, can be stated as:

Theorem 4: *In coalitional NTU economy (14) without transaction costs, owners will produce the efficient payoffs by allocating inputs to efficient firms based on firms' shadow values, and will choose an efficient payoff vector from the non-empty NTU core.*

Precisely, the efficient payoffs, core payoffs and efficient firms are given in (18), (25) and (26), respectively. Analogous to the TU case, the strong conclusion of NTU Coase theorem results from the advantages of utilizing generated-payoffs: in the usual NTU core, players just choose from $\partial V(N)$; whereas in the newly defined NTU core, players choose from the game's efficient payoffs, which in general are better than $\partial V(N)$.

6. Conclusion and discussion

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[6] Keep in mind that efficient payoffs here are only weakly efficient. The payoff *(2, 1, 2)* is only weakly efficient, it is not efficient in the sense of Pareto because it is Pareto-dominated by *(2, 2, 2)*.

The above analysis has explored the possibility that owners in a coalitional economy sometimes could produce better payoffs than the monopoly payoff. It has revealed that a firm's shadow value is equal to its balancing weight and has represented the major aspects and main insights of the Coase theorem by two mathematical theorems in coalitional economies.

By modeling non-market allocation of resources as a coalitional economy or bargaining economy, the paper not only has advanced coalition formation from partitions to minimal balanced collections but also has advanced the study of the Coase theorem in three areas. First, our two versions of the Coase theorem (i.e., Theorems 2 and 4) show that it is sometimes socially optimal for firms to shut down parts of their operations and for workers to have two or more part-time jobs. This conclusion, previously unreported, will provide a new line of argument in studying labor theory, production theory, and other related fields in economics.

Second, our two versions show precisely how the size of transaction costs in each merger or coalition could prevent or allow its formation, and this provides a two-step procedure for empirically estimating the size of transaction costs involved in each previous or future application of the Coase theorem: 1) identify the merger (or the parties in the transaction problem under investigation) and convert it into a coalitional economy, and 2) compute its minimum no-blocking payoff (*mnbp*). The difference between the merger's payoff and its *mnbp* is the estimated upper (lower) bound of transaction costs below (above) which the optimal outcome predicted by the Coase theorem holds (fails).

Finally, our two versions have the potential to open doors for applying the Coase theorem to an endless range of future studies, not only to both TU and NTU transaction problems but also to all non-transaction problems that are modeled by coalitional games.

Appendix

Proof of Theorem 1: For each $S \neq N$, let $e_S = (x_1, \ldots, x_n)' \in \boldsymbol{R}^+_n$ be its incidence vector or the column vector such that $x_i = 1$ if $i \in S$ and $x_i = 0$ if $i \notin S$, and $e = e_N = (1, \ldots, 1)'$ be a column vector of ones. Then, the dual problem for the minimization problem (8) is the following maximization problem:

(27) $$Max\ \{\Sigma_{S \neq N} y_S\, v(S) \mid y_S \geq 0 \text{ for all } S \neq N; \text{ and } \Sigma_{S \neq N} y_S\, e_S \leq e\}.$$

We will show that (27) is equivalent to the maximization problem (6). First, we show that the inequality constraints in (27) can be replaced by equation constraints.

Let $Ay \leq e$ and $y \geq 0$ denote the constraints in (27), where $A = A_{n \times (2^n - 2)} = [e_S \mid S \neq N]$ is the constraint matrix, and y is the (2^n-2) dimensional vector whose indices are the proper coalitions. Let the rows of A be $a_1, \ldots, a_n$, and for each feasible y, let $T = T(y) = \{i \mid a_i \cdot y < 1\}$ be the set of loose constraints, so $N \backslash T = \{i \mid a_i \cdot y = 1\}$ is the set of binding constraints.

If $T(y) \neq \emptyset$, let z be defined as: $z_S = y_S + (1 - a_i \cdot y)$ if $S = \{i\}$, for each $i \in T$, and $z_S = y_S$ if $S \neq \{i\}$ for all $i \in T$. One sees that $z > y$ and $T(z) = \emptyset$. Hence, for any y with $T(y) \neq \emptyset$, there exists $z \geq 0$, $Az = e$ such that $\Sigma_{S \neq N} y_S\, v(S) \leq \Sigma_{S \neq N} z_S\, v(S)$. Hence, the feasible set of (27) can be reduced to $\{z \mid z \geq 0, Az = e\}$, without affecting the maximum value. So the maximization problem in (27) is equivalent to the following problem:

(28) $$Max\ \{\Sigma_{S \neq N} y_S\, v(S) \mid Ay = e, \text{ and } y \geq 0\}.$$

Next, we establish the one-to-one relationship between the extreme points of (28) and the minimal balanced collections. Note that for each feasible y in (28), $\mathcal{B}(y) = \{S \mid y_S > 0\}$ is a balanced collection. Let y be an extreme point of (28). We now show that $\mathcal{B}(y) = \{S \mid y_S > 0\}$ is a minimal balanced collection.

Assume by way of contradiction that $\mathcal{B}(y)$ is not minimal, then there exists a balanced subcollection $B' \subset \mathcal{B}(y)$ with balancing vector z. Note that $z_S > 0$ implies $y_S > 0$. Therefore, for a sufficiently small $t > 0$ (e.g., $0 < t \leq ½$, and $t \leq \text{Min}\ \{y_S / |z_S - y_S| \mid \text{all S with } y_S \neq z_S\}$), one has

$$w = y - t(y-z) \geq 0,\ w' = y + t(y-z) \geq 0.$$

$Ay = e$ and $Az = e$ lead to $Aw = e$ and $Aw' = e$. But $y = (w+w')/2$ and $w \neq w'$ contradict the assumption that y is an extreme point. So $\mathcal{B}(y)$ must be minimal.

Now, let $\mathcal{B} = \{T_1, \ldots, T_k\}$ be a minimal balanced collection with a balancing vector z.

We need to show that z is an extreme point of (28). Assume again by way of contradiction that z is not an extreme point, so there exists $w \neq w'$ such that $z = (w+w')/2$. By $w \geq 0$, $w' \geq 0$, one has

$$\{S \mid w_S > 0\} \subseteq \mathcal{B} = \{S \mid z_S > 0\}, \text{ and } \{S \mid w'_S > 0\} \subseteq \mathcal{B} = \{S \mid z_S > 0\}.$$

The above two expressions show that both w and w' are balancing vectors for some subcollections of $\mathcal{B}$. Because $\mathcal{B}$ is minimal, one has $w = w' = z$, which contradicts $w \neq w'$. Therefore, z must be an extreme point of (28).

Finally, by the standard results in linear programming, the maximal value of (28) is achieved among the set of its extreme points, which are equivalent to the set of the minimal balanced collections, so (28) is equivalent to $Max\ \{\Sigma_{S\in\mathcal{B}} y_S v(S)\}$, subject to the requirements that $N \notin \mathcal{B}$ and $\mathcal{B}$ is a minimal balanced collection with the balancing vector y. This shows that (27) is equivalent to the maximization problem (6) for *mgp*, which completes the proof for Theorem 1. **Q.E.D**

Proof of Lemma 1: Given Theorem 1, it is straightforward to show parts (i-iii). Note that part (i) was first proved using $\text{Min}\{\Sigma_{i\in N} x_i \mid x \in X(v(N)),\ \Sigma_{i\in S} x_i \geq v(S), \text{ all } S \subseteq N\}$. **Q.E.D**

Proof of Theorem 2: Discussions before the theorem serve as a proof. **Q.E.D**

Our proof for Theorem 3 uses the following lemma on open covering of the simplex $\Delta^N = X(1) = \{x \in \mathbf{R}^+_n \mid \Sigma_{i\in N} x_i = 1\}$.

Lemma 3 (Scarf 1967a, Zhou 1994)**:** *Let $\{C_S\}$, $S \neq N$, be a family of open subsets of Δ^N that satisfy $\Delta^{N\setminus\{i\}} = \{x \in \Delta^N \mid x_i = 0\} \subset C_{\{i\}}$ for all $i \in N$, and $\cup_{S\neq N} C_S = \Delta^N$, then there exists a balanced collection of coalitions $\mathcal{B}$ such that $\cap_{S\in\mathcal{B}} C_S \neq \emptyset$.*

Proof of Theorem 3: Let *UBP* be the set of unblocked payoffs in (19), and *EGP* be the boundary or (weakly) efficient set of the generated payoff in (17). We shall first show that $UBP \cap EGP \neq \emptyset$.

For each coalition $S \neq N$, let $W_S = \{Int\, V(S) \times \mathbf{R}^{-S}\} \cap EGP$ be an open (relatively in *EGP*) subset of *EGP*, where $Int\, V(S) = V(S) \setminus \partial V(S)$ is the interior of $V(S)$. For each minimal balanced collection of coalitions $\mathcal{B}$, we claim that

(29) $\cap_{S\in\mathcal{B}} W_S = \varnothing$

holds. If (29) is false, there exists $y\in EGP$ and $y\in Int\, V(S)\times\boldsymbol{R}^{-S}$ for each $S\in\mathcal{B}$. We can now find a small $t>0$ such that $y+te\in Int\, V(S)\times\boldsymbol{R}^{-S}$ for each $S\in\mathcal{B}$, where e is the vector of ones. By the definition of generated payoffs in (17), $y+te\in GP(\mathcal{B}) = \cap_{S\in\mathcal{B}}\{V(S)\times\boldsymbol{R}^{-S}\}\subset GP$, which contradicts $y\in EGP$. This proves (29).

Now, suppose by way of contradiction that $UBP\cap EGP = \varnothing$. Then, $EGP\subset UBP^C$, where superscript C denotes the complement of a set. The definition of W_S and

$$UBP^C = \{\cap_{S\neq N}\{[V(S)\backslash\partial V(S)]^C\times\boldsymbol{R}^{-S}\}\}^C = \cup_{S\neq N}\{Int\, V(S)\times\boldsymbol{R}^{-S}\}$$

together lead to $\cup_{S\neq N} W_S = EGP$, so $\{W_S\}$, $S\neq N$, is an open cover of EGP.

Because the set of generated payoffs is comprehensive and bounded from above, and the origin is in its interior (by $\partial V(i)>0$, all i), the following mapping from EGP to Δ^N:

$$f: x \rightarrow x/\Sigma x_i,$$

is a homeomorphism. Define $C_S = f(W_S)$ for all $S\subseteq N$, one sees that $\{C_S\}$, $S\neq N$, is an open cover of $\Delta^N = f(EGP)$.

For each $i\in N$, $\partial V(i)>0$ leads to $EGP\cap\{x\in\boldsymbol{R}^n \mid x_i = 0\}\subset W_{\{i\}}$, which in turn leads to $\Delta^{N\backslash\{i\}} = \{x\in\Delta^N \mid x_i = 0\} = f(EGP\cap\{x\in\boldsymbol{R}^n \mid x_i = 0\})\subset C_{\{i\}} = f(W_{\{i\}})$. Therefore, $\{C_S\}$, $S\neq N$, is an open cover of Δ^N satisfying the conditions of Scarf-Zhou open covering theorem, so there exists a balanced collection of coalitions $\mathcal{B}_0$ such that

$$\cap_{S\in\mathcal{B}_0} C_S \neq \varnothing, \text{ or } \cap_{S\in\mathcal{B}_0} W_S \neq \varnothing,$$

which contradicts (29). Hence, $UBP\cap EGP \neq \varnothing$.

For each $x\in UBP\cap EGP$, we claim $x\in MNBF$. If this is false, we can find a small $\tau>0$ such that $x-\tau e\in UBP$. Let $\mathcal{B}\in B$ be the minimal balanced collection of coalitions such that $x\in GP(\mathcal{B}) = \cap_{S\in\mathcal{B}}\{V(S)\times\boldsymbol{R}^{-S}\}$. Then, $x-\tau e\in Int\, V(S)\times\boldsymbol{R}^{-S}$ for each $S\in\mathcal{B}$, which contradicts $x-\tau e\in UBP$. Therefore, $MNBF\cap EGP = UBP\cap EGP\neq\varnothing$. **Q.E.D**

Proof of Lemma 2 and Theorem 4: The discussions preceding the lemma and theorem serve as their proofs. **Q.E.D**

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