On the Theory of Effective Demand Under Stochastic Rationing

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ON THE THEORY OF EFFECTIVE DEMAND UNDER STOCHASTIC RATIONING.*

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The properties of effective demands in a non-market clearing environment can be analyzed either in a deterministic framework or in a stochastic framework. It was observed that the existing deterministic models cannot yield simultaneously a reliable measure of disequilibrium and a satisfactory theory of individual behaviour. (See Grandmont (1982)). The theory of effective demand under stochastic rationing has hence received much attention in the recent work of Gale (1979), Green (1980), Svensson (1980) and Weinrich (1984). One of the fundamental issues in the theory of effective demand under stochastic rationing is whether or not the stochastic rationing scheme is manipulable. Green (1980) considers the case in which the random realisation of the actual transaction for each agent depends only on his own effective demand and the aggregate effective demand and supply. With other assumptions such as anonymity, voluntary trade, continuity, and feasibility, Green has shown that the expectation of actual transaction must be linear in the agent's own effective demand. In other words, to be consistent with these assumptions the stochastic rationing scheme must be manipulable.

The purpose of this study is to explore the generality of this linearity result. It will be shown that the linearity result holds with fewer assumptions than in Green (1980): the anonymity assumption can be dispensed with and the voluntary trade and continuity assumptions can be weakened. Furthermore, in contrast to Green (1980) which provides only sufficient conditions, we will find a set of necessary and sufficient conditions for the expectation of stochastic rationing scheme to be linear in the agent's effective demand.

I. THE STOCHASTIC RATIONING SCHEME

Consider an economy with I agents who have effective demands z_i, $i = 1, \dots I$ in a single market. Aggregate effective demand and supply are given by $Z^+ = \Sigma_{i=1}^I \max(z_i, 0)$ and $Z^- = \Sigma_{i=1}^I \min(z_i, 0)$. The general stochastic rationing scheme is a random function which maps the vector of effective demands of all agents into actual transactions x_i (see Gale (1979)):

$$x_i = \phi_i(\mathbf{z}), \text{ where } \mathbf{z} = (z_1, \dots, z_I). \qquad (1)$$

The special rationing scheme considered by Green (1980) is as follows:

A(0) $x_i = \phi_i(z_i, Z^+, Z^-)$,

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satisfying

A (i) $|\phi_i| \leqslant |z_i|$ with probability one,

A (ii) $z_i \phi_i \geqslant 0$ with probability one,

A (iii) $\Sigma_{i=1}^{I} \mathrm{E}\phi_i(\cdot) = 0$ for all effective demand vectors $\mathbf{z} = (z_1, \ldots, z_I)$,

A (iv) The mean $\bar{\phi}_i(\cdot) = \mathrm{E}\phi_i(\cdot)$ is the same for all $i = 1, \ldots, I$,

A (v) The mean $\bar{\phi}_i(\cdot)$ is continuous in z_i,

A (vi) $z_i \cdot (Z^- + Z^+) < 0$ implies $\phi_i = z_i$.

Conditions A (i) and A (ii) embody the requirement of voluntary trade. Condition A (iii) is the feasibility condition, condition A (iv) the anonymity assumption, condition A (v) the continuity assumption, and condition A (vi) the 'short sided' trading rule. The set of conditions A (0)–A (v) is used by Green to show that $\bar{\phi}$ is linear in z_i.

Here we will prove the same result with much weaker conditions. Conditions A (ii) and A (iv) will be disregarded and conditions A (i) and A (v) will be replaced by weaker conditions A (i′) and A (v′):

A (i′) If $z_i = 0$, then $\phi_i = 0$ with probability one.

A (v′) $\bar{\phi}_i(\cdot)$ is continuous at a single point z_i^0.

Condition A (i′) states that zero trade offers lead to zero transactions, which is implied by, but weaker than, A (i). Condition A (v′) replaces continuity at every point with continuity at a single point. We will show that A (0), A (i′), A (iii) and A (v′) are sufficient for $\bar{\phi}_i$ to be linear in z_i.

The existence of weaker conditions motivates the search for a minimal set of sufficient conditions for the linearity result. In particular, can we relax A (0) and still get linearity? The answer is negative. We will show that in a wider class of rationing schemes the necessary condition for linearity is exactly the special form A (0). Since the maintained assumptions such as A (iii), A (iv) and A (vi) are mild restrictions which are usually imposed on rationing schemes in the literature, our result may demonstrate that assuming the special form A (0) of rationing schemes is almost equivalent to requiring linearity of rationing schemes.

In this paper we will search for necessary conditions of linearity in the following class A (0′) of rationing schemes. For any vector **z** of effective demands the index is ordered such that $z_i \geqslant 0$ for $1 \leqslant i \leqslant m$ and $z_i < 0$ for $m+1 \leqslant i \leqslant I$.

A (0′) $x_i = \phi_i[z_i, g_1(\mathbf{z}^+), g_2(\mathbf{z}^{-1})]$,

where

$$\mathbf{z}^+ = (z_1, \ldots, z_m) \quad \text{with} \quad z_i \geqslant 0 \quad \text{for all} \quad 1 \leqslant i \leqslant m,$$

$$\mathbf{z}^- = (z_{m+1}, \ldots, z_I) \quad \text{with} \quad z_i < 0 \quad \text{for all} \quad m+1 \leqslant i \leqslant I,$$

and $\mathbf{g}_k = (g_{k1}, \ldots, g_{kn_k})$ $(k = 1, 2)$ is a vector of symmetric functions of the arguments.

Notice that in A(0′) $g_{kl_k} (l_k = 1, \ldots, n_k)$ can be any symmetric deterministic or random function. Condition A(0) is a special case of A(0′):

$$\text{A}(0)\quad g_1(\mathbf{z}^+) = \sum_{i=1}^{m} z_i = Z^+,$$

$$g_2(\mathbf{z}^-) = \sum_{i=m+1}^{I} z_i = Z^- \quad \text{and} \quad n_k = 1 \quad (k = 1, 2).$$

The rationing scheme A(0′) is different from the general rationing scheme $x_i = \phi_i(\mathbf{z})$ of (1) in separating the vector $\mathbf{z}$ into two parts $\mathbf{z}^+$ and $\mathbf{z}$. However, it is still quite general. In the following example of random matching adopted from Green (1980, Example 1) with $I = 4$, the rationing scheme satisfies A(0′) but not A(0).

Example. The rationing scheme places the agents in a random order and pairs effective demands and supplies at each step of this sequence. The random order induces a distribution of final trades for any given vector of effective demands. Suppose there are four agents with $z_1 > 0$, $z_2 > 0$, $z_3 > 0$, $z_4 < 0$, $z_1 + z_2 + z_3 = 3$ and $z_4 = -2$. Consider the following two cases:

(A) $z_1 = 1$, $z_2 = 1$, $z_3 = 1$,

(B) $z_1 = 1$, $z_2 = 2$, $z_3 = 0$.

Suppose this scheme is described by A(0), we must have

$$\mathrm{E}\phi_1(1, 3, -2) = \tfrac{2}{3} \quad \text{in case } A.$$

and

$$\mathrm{E}\phi_1(1, 3, -2) = \tfrac{1}{2} \quad \text{in case } B.$$

This means that A(0) is not a good description of the scheme of random matching. (However, we cannot show this with $I = 3$ as in Green's example.) Since this rationing scheme depends on the distribution of effective demands, it can be described by A(0′):

$$\mathrm{E}\phi_1[1, g_1(1, 1, 1), g_2(-2)] = \tfrac{2}{3} \quad \text{in case } A$$

and

$$\mathrm{E}\phi_1[1, g_1(1, 2, 0), g_2(-2)] = \tfrac{1}{2} \quad \text{in case } B,$$

where g_1 can be a summary statistic, such as the variance of the distribution of effective demands. Notice also that this rationing scheme is not linear in the agent's effective demand:

$$\mathrm{E}\phi_1\,[2, g_1(1, 2, 0), g_2(-2)] = \tfrac{3}{2}.$$

Finally, for the rationing scheme A(0′) the expected final transaction is linear in the agent's effective demand if the following is satisfied:

$$(L)\ \bar{\phi}_i[z_i, g_1(\mathbf{z}^+), g_2(\mathbf{z}^-)] = z_i S_i^+[g_1(\mathbf{z}^+), g_2(\mathbf{z}^-)] \quad \text{for} \quad z_i \geqslant 0,$$

$$= z_i S_i^-[g_1(\mathbf{z}^+), g_2(\mathbf{z}^-)] \quad \text{for} \quad z_i < 0.$$

Linearity (L) means that $\bar{\phi}_i$ is linear in its first argument, over the positive and negative half-lines, but perhaps with different slopes.

II. RESULTS

THEOREM 1. *Let $I \geqslant 4$. Given conditions $A\,(i')$, $A\,(iii)$ and $A\,(v')$, the special form of $A\,(0)$ implies linearity (L) in the following form:*

$$\begin{aligned}\bar{\phi}_i(z_i, Z^+, Z^-) &= z_i S^+(Z^+, Z^-) \quad \text{for} \quad z_i \geqslant 0,\\ &= z_i S^-(Z^+, Z^-) \quad \text{for} \quad z_i < 0. \end{aligned} \tag{2}$$

Proof. The first step is to prove anonymity from A(0), A(i′) and A(iii). By A(0) and A(iii), a permutation of z_1 and z_i leaves the sum $\Sigma_{i=1}^{I} \bar{\phi}_i(z_i, Z^+, Z^-) = 0$ and $I-2$ terms unchanged. Thus

$$\bar{\phi}_1(z_1, Z^+, Z^-) + \bar{\phi}_i(z_i, Z^+, Z^-) = \bar{\phi}_1(z_i, Z^+, Z^-) + \bar{\phi}_i(z_1, Z^+, Z^-), \tag{3}$$

which may be rearranged as

$$\bar{\phi}_1(z_1, Z^+, Z^-) - \bar{\phi}_i(z_1, Z^+, Z^-) = \bar{\phi}_1(z_i, Z^+, Z^-) - \bar{\phi}_i(z_i, Z^+, Z^-), \tag{4}$$

which must hold for any z_1 and z_i. This implies that $\bar{\phi}_1(z, Z^+, Z^-) - \bar{\phi}_i(z, Z^+, Z^-)$ does not depend on its first argument z and must be equal to a function independent of z:

$$\bar{\phi}_1(z, Z^+, Z^-) - \bar{\phi}_i(z, Z^+, Z^-) = K_i(Z^+, Z^-)$$

or

$$\bar{\phi}_i(z, Z^+, Z^-) = \bar{\phi}_1(z, Z^+, Z^-) - K_i(Z^+, Z^-) \quad \text{for all } z \text{ and } i. \tag{5}$$

Since (5) is true for $z = 0$, by A(i′), we have $K_i = 0$ and

$$\bar{\phi}_i(z, Z^+, Z^-) = \bar{\phi}(z, Z^+, Z^-) \quad \text{for all } z \text{ and } i.$$

The next step is to prove linearity. To avoid trivial cases, we consider a vector $\mathbf{z} = (z_1, z_2, z_3, \ldots, z_I)$ with $z_1 > 0$, $z_2 > 0, z_3 > 0$ and $z_{m+1} < 0$. By condition A(iii) we have

$$\begin{aligned}\bar{\phi}(z_1, Z^+, Z^-) + \bar{\phi}(z_2, Z^+, Z^-) + \bar{\phi}(z_3, Z^+, Z^-) &= \bar{\phi}(z_1 + z_2, Z^+, Z^-)\\ &\quad + \bar{\phi}(0, Z^+, Z^-) + \bar{\phi}(z_3, Z^+, Z^-).\end{aligned}$$

We can choose z_3 to make Z^+ unchanged while varying z_1 and z_2 in a closed interval. (That is the reason for requiring $I \geqslant 4$, also see Weinrich (1982).) From A(i′), $\bar{\phi}(0, Z^+, Z^-) = 0$, thus

$$\bar{\phi}(z_1, Z^+, Z^-) + \bar{\phi}(z_2, Z^+, Z^-) = \bar{\phi}(z_1 + z_2, Z^+, Z^-), \tag{7}$$

where Z^+, Z^- are fixed for changing z_1 and z_2.

Equation (7) is Cauchy's functional equation. (For the theory and applications of functional equations, see Aczel (1966) and Lau (1982, Appendix A).) If A(v′) is satisfied, then

$$\bar{\phi}(z_i, Z^+, Z^-) = z_i S^+(Z^+, Z^-) \quad \text{for real} \quad z_i \geqslant 0.$$

Similar arguments can be presented for $z_i < 0$. Q.E.D.

COROLLARY 1. *Let* $I \geqslant 4$. *If the rationing scheme satisfies* $A(0)$, $A(i')$, $A(iii)$, $A(v')$ *and* $A(vi)$, *then*

$$\begin{aligned}\bar{\phi}_i(z_i, Z^+, Z^-) &= z_i \cdot \min\{-Z^-/Z^+, 1\} \quad \text{if} \quad z_i \geqslant 0, \\ &= z_i \cdot \min\{-Z^+/Z^-, 1\} \quad \text{if} \quad z_i < 0.\end{aligned} \tag{8}$$

Proof: By condition A(iii) and Theorem 1, $S^+(Z^+, Z^-)Z^+ = -S^-(Z^+, Z^-)Z^-$. Consider $Z^+ > -Z^-$, the short sided rule A(vi) implies

$$S^-(Z^+, Z^-) = 1 \quad \text{and} \quad S^+(Z^+, Z^-) = -Z^-/Z^+.$$

The proof for the case $Z^+ \leqslant -Z^-$ is similar. Q.E.D.

THEOREM 2. *Let* $I \geqslant 4$. *Given conditions* $A(0')$, $A(iii)$, $A(iv)$ *and* $A(vi)$, *linearity* (L) *implies that the stochastic rationing scheme must be of the form* $A(0)$.

Proof: From Linearity, feasibility (A(iii)) and anonymity (A(iv)) we have

$$S^+[g_1(\mathbf{z}^+), g_2(\mathbf{z}^-)] \sum_{i=1}^{m} z_i + S^-[g_1(\mathbf{z}^+), g_2(\mathbf{z}^-)] \sum_{i=m+1}^{I} z_i = 0. \tag{9}$$

Consider the case $Z^+ > -Z^-$, the short sided rule (A(vi)) implies that $S^-(\cdot) = 1$. Therefore,

$$S^+[g_1(z_1, \ldots, z_m), g_2(z_{m+1}, \ldots z_I)] = -Z^-/Z^+. \tag{10}$$

The function $S^+(\cdot)$ depends on the vector $\mathbf{z}^+$ only through the sum of its components, Z^+. Thus

$$g_1(z_1, \ldots, z_m) = \sum_{i=1}^{m} z_i = Z^+.$$

Similarly we can show that

$$g_2(z_{m+1}, \ldots, Z_I) = \sum_{i=m+1}^{I} z_i = Z^-.$$

Therefore, the rationing scheme must be of the form

$$x_i = \phi(z_i, Z^+, Z^-).$$

Q.E.D.

Notice that Theorem 2 is proved with the anonymity assumption and the short sided rule. By taking the union of two sets of maintained assumptions in Theorem 1 and 2, we bring the two parts together into a final result.

THEOREM 3. *Let* $I \geqslant 4$. *Assume that the stochastic rationing scheme* $A(0')$ *satisfies condition* $A(i')$, $A(iv)$, $A(v')$ *and* $A(vi)$. *The expected final trade is linear in the agent's effective demand as in* (L) *if and only if the rationing scheme depends only on the agent's effective demand and the aggregate effective demand and supply as in* $A(0)$.

III. CONCLUSION

This note has shown that the linearity property of the stochastic rationing scheme can be obtained with fewer assumptions than in Green (1980). In the class A(o′) of rationing schemes the dependence of rationing outcomes on aggregates alone (condition A(o)) is shown to be indispensible for this linearity result.

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