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Exact aggregation under summability and homogeneity with individually variable prices

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Abstract

Exact aggregation of a system of individual expenditure functions with a single, individually variable price is analyzed. It is shown, under summability and homogeneity, that the individual and aggregate expenditure functions must take one of two specific functional forms.

Keywords: Exact aggregation; Variable prices; Homogeneity; Summability; Functional form

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1. Introduction

Most studies of aggregation of individual demand functions, such as Gorman (1953), Muelbauer (1975, 1976), Jorgenson et al. (1982) and Lau (1982a,b), assume that all individual consumers face the same price vector. In contrast, Lau and Wu (1987, 1993) study exact aggregation of individual demand functions when the price of one commodity is allowed to vary across individuals. There are many reasons, such as a difference in transportation costs and search costs, that may cause the price of some commodity to be different across individuals. Lau and Wu (1987, 1993) demonstrate that exact aggregability *and* summability of a system of individual demand functions imply that the individual demands on the commodity with an individually variable price must be independent of all other prices. It was therefore concluded that the demand function approach is not satisfactory and that the expenditure function approach should be used.

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In this paper we adopt the expenditure function approach to study exact aggregation with individually variable prices. We explore the implications of imposing the additional restriction of homogeneity of degree one in all prices and total expenditure on the individual expenditure functions, i.e. the restriction of 'no money illusion', implied by individual utility maximization. We show that exact aggregability, summability and homogeneity of a system of individual expenditure functions imply that the expenditure functions must take one of two specific functional forms.

2. The expenditure function approach

Consider an economy with N individual consumers, indexed by $i = 1, \ldots, N$, and $(m+1)$ distinct commodities, indexed by $j = 1, \ldots, m+1$. Individuals face the same price vector p for the first m commodities. However, for the $(m+1)$th commodity, the ith individual faces the price w_i, $i = 1, \ldots, N$. The expenditure of the ith individual on the jth commodity is given by a real-valued function $e_{ij}(p, w_i, M_i)$, where M_i is the ith individual's total expenditure. Aggregate expenditure on the jth commodity is the sum of individual expenditures $\Sigma_{i=1}^{N} e_{ij}(p, w_i, M_i)$. Following Lau and Wu (1987, 1993) we adopt a generalized concept of an aggregate expenditure function $E_j(\cdot)$ as a function of real-valued functions, $g_k(\cdot)$, of individual total expenditures and individually variable prices, called index functions:

$$\begin{aligned}&\sum_{i=1}^{N} e_{ij}(p, w_i, M_i)\\ &\quad = E_j(p, g_1(w_1, \ldots, w_N, M_1, \ldots, M_n), \ldots, g_n(w_1, \ldots, w_N, M_1, \ldots, M_N)), \qquad (1)\\ &\quad n < N, \quad j = 1, \ldots, m+1, \quad p \in R_+^m, \quad w_i \in R_+, \quad M_i \in \bar{R}_+ .\end{aligned}$$

A collection of N systems of individual expenditure functions, $e_{ij}(p, w_i, M_i)$, is said to be *exactly aggregable* if there exists a system of aggregate expenditure functions, $E_j(\cdot)$, and index functions, $g_k(\cdot)$, such that Eq. (1) holds as an identity. By using the Fundamental Theorem of Exact Aggregation (Theorem 2 of Lau, 1982b), and the results of Lau and Wu (1987, 1993), it is straightforward to establish the following:

Theorem 1. A system of aggregate expenditure functions can be written in the form of Eq. (1), *where*

(1) *each index function $g_k(\cdot)$ is non-constant and symmetric with respect to the subscripts 1–N and non-separable with respect to w_i and M_i;*

(2) *there does not exist any functional relationship among the $g_k(\cdot)$;*

(3) *there exist price vectors $p^1, \ldots, p^n$ such that a system of n functions of the type $E_j(p^k, g_1, \ldots, g_n)$, $k = 1, \ldots, n$, is invertible in $g_1, \ldots, g_n$, and the range of each such $E_j(p^k, g_1, \ldots, g_n)$ is an interval of the non-negative real line, $k = 1, \ldots, n$;*

(4) *each $E_j(p, g_1, \ldots, g_n)$ is non-negative only if*

(a) $e_{ij}(p, w_i, M_i) = \sum_{k=1}^{n} h_{jk}(p) g_k^*(w_i, M_i) + k_{ij}^*(p)\,, \quad i = 1, \ldots, N\,, \quad j = 1, \ldots, m+1\,,$

and

(b) $E_j(p, g_1(w_1, \ldots, w_N, M_1, \ldots, M_N), \ldots, g_n(w_1, \ldots, w_N, M_1, \ldots, M_N))$

$$= \sum_{k=1}^{n} h_{jk}(p)\left[\sum_{i=1}^{N} g_k^*(w_i, M_i)\right] + \sum_{i=1}^{N} k_{ij}^*(p)\,, \quad j = 1, \ldots, m+1\,,$$

where the $h_{jk}(p)$ are arbitrary real-valued functions, the $g_k^*(w_i, M_i)$ are a set of arbitrary linearly independent real-valued functions, and the $k_{ij}^*(p)$ are arbitrary real-valued functions.

If, in addition, it is assumed that the aggregate expenditure functions, $E_j(\cdot)$, $j = 1, \ldots, m+1$, are zero when aggregate total expenditure $\Sigma_{i=1}^{N} M_i$ is zero and that the individual expenditure functions, $e_{ij}(\cdot)$, are non-negative, then the $e_{ij}(\cdot)$ must take the form:

$$e_{ij}(p, w_i, M_i) = \sum_{k=1}^{n} h_{jk}(p) g_k^{**}(w_i, M_i)\,, \quad i = 1, \ldots, N\,, \quad j = 1, \ldots, n+1\,, \tag{2}$$

where $g_k^{**}(w_i, M_i) \equiv g_k^*(w_i, M_i) - g_k^*(w_i, 0)$, $k = 1, \ldots, n$. In other words, the $g_k^{**}(w_i, M_i)$ can be taken so that $g_k^{**}(w_i, 0) = 0$, $k = 1, \ldots, n$.

3. The implications of summability

First, we consider the implications of summability. A system of individual expenditure functions is said to be summable if the sum of expenditures on the different commodities is equal to the total expenditure:

$$\sum_{j=1}^{m+1} e_{ij}(p, w_i, M_i) = \sum_{j=1}^{m+1} \sum_{k=1}^{n} h_{jk}(p) g_k^{**}(w_i, M_i) = M_i\,, \quad i = 1, \ldots, N\,, \quad p \in R_+^m\,,$$
$$w_i \in R_+\,, \quad M_i \in \bar{R}_+\,. \tag{3}$$

Let p^1 and p^2 be two arbitrary positive price vectors. Then summability implies

$$\sum_{j=1}^{m+1} \sum_{k=1}^{n} h_{jk}(p^1) g_k^{**}(w_i, M_i) = \sum_{j=1}^{m+1} \sum_{k=1}^{n} h_{jk}(p^2) g_k^{**}(w_i, M_i)\,, \tag{4}$$

so that

$$\sum_{k=1}^{n} \sum_{j=1}^{m+1} [h_{jk}(p^1) - h_{jk}(p^2)] g_k^{**}(w_i, M_i) = 0\,. \tag{5}$$

Linear independence of the index functions, $g_k^{**}(\cdot)$, then implies that $h_k^*(p) \equiv \Sigma_{j=1}^{m+1} h_{jk}(p) = C_k$, $k = 1, \ldots, n$, where the C_k terms are constants. Eq. (3) then implies

$$\sum_{k=1}^{n} C_k g_k^{**}(w_i, M_i) = M_i\,. \tag{6}$$

Without loss of generality, we may set

$$C_1 = \sum_{j=1}^{m+1} h_{j1}(p) = 1$$

and

$$g_1^{**}(w_i, M_i) = M_i - \sum_{k=2}^{n} C_k g_k^{**}(w_i, M_i) . \tag{7}$$

The individual expenditure functions may be rewritten as

$$\begin{aligned} e_{ij}(p, w_i, M_i) &= h_{j1}(p)M_i + \sum_{k=2}^{n} [h_{jk}(p) - C_k h_{j1}(p)] g_k^{**}(w_i, M_i) \\ &= h_{j1}(p)M_i + \sum_{k=2}^{n} h_{jk}^*(p) g_k^{**}(w_i, M_i) \end{aligned} \tag{8}$$

where

$$h_{jk}^*(p) \equiv h_{jk}(p) - C_k h_{j1}(p) , \quad j = 1, \ldots, m+1 , \quad k = 1, \ldots, n , \tag{9}$$

$$\sum_{j=1}^{m+1} h_{j1}(p) = 1 , \tag{10}$$

$$\sum_{j=1}^{m+1} h_{jk}^*(p) = \sum_{j=1}^{m+1} h_{jk}(p) - C_k \sum_{j=1}^{m+1} h_{j1}(p) = C_k - C_k = 0 , \quad k = 2, \ldots, n . \tag{11}$$

4. The implications of summability and homogeneity

In this section we assume that the individual expenditure functions are once continuously differentiable and analyze the case of two index functions ($n = 2$). We consider the implications of homogeneity of degree one of the individual expenditure functions in all prices and total expenditure in addition to summability. Homogeneity of degree one implies:

$$\lambda h_{j1}(p)M_1 + \lambda h_{j2}^*(p) g_2^{**}(w_i, M_i) = h_{j1}(\lambda p)\lambda M_i + h_{j2}^*(\lambda p) g_2^{**}(\lambda w_i, \lambda M_i) . \tag{12}$$

Dividing both sides by λM_i and defining $\ell(w_i, M_i) \equiv g_2^{**}(w_i, M_i)/M_i$, Eq. (12) may be rewritten as

$$h_{j1}(p) + h_{j2}^*(p)\ell(w_i, M_i) = h_{j1}(\lambda p) + h_{j2}^*(\lambda p)\ell(\lambda w_i, \lambda M_i) . \tag{13}$$

Let w_i and M_i be set equal to w_{i1} and M_{i1}, and w_{i2} and M_{i2} respectively. Then we have

$$h_{j2}^*(p)[\ell(w_{i1}, M_{i1}) - \ell(w_{i2}, M_{i2})] = h_{j2}^*(\lambda p)[\ell(\lambda w_{i1}, \lambda M_{i1}) - \ell(\lambda w_{i2}, \lambda M_{i2})] , \tag{14}$$

which implies that

$$h_{j2}^*(\lambda p) = \ell^*(\lambda) h_{j2}^*(p) . \tag{15}$$

This is the generalized Euler equation with the well-known solution of $\ell^*(\lambda) = \lambda^{-\sigma}$. Hence, $h_{j2}^*(p)$ is a homogeneous function of degree σ. Substituting this back into Eq. (13), we have:

$$\ell(\lambda w_i, \lambda M_i) - \lambda^\sigma \ell(w_i, M_i) = \lambda^\sigma \left[\frac{h_{j1}(p) - h_{j1}(\lambda p)}{h_{j2}^*(p)} \right] . \tag{16}$$

This equation implies that each side is at most a function of the common variable λ, say $k(\lambda)$:

$$\lambda^\sigma \left[\frac{h_{j1}(p) - h_{j1}(\lambda p)}{h_{j2}^*(p)} \right] = k(\lambda) . \tag{17}$$

Next, we consider two cases: $\sigma = 0$ and $\sigma \neq 0$. If $\sigma = 0$, then Eq. (16) becomes:

$$\ell(\lambda w_i, \lambda M_i) - \ell(w_i, M_i) = k(\lambda) ,$$

or

$$\exp[\ell(\lambda w_i, \lambda M_i)] = \exp[k(\lambda)] \exp[\ell(w_i, M_i)] .$$

This equation can be written as

$$G_2(\lambda w_i, \lambda M_i) = \exp[k(\lambda)]\, G_2(w_i, M_i) = f(\lambda) G_2(w_i, M_i) , \tag{18}$$

with the solution $f(\lambda) = \lambda^\mu$, just as in Eq. (15). Equivalently, we have the following expression:

$$\ell(w_i, M_i) \equiv \frac{g_2^{**}(w_i, M_i)}{M_i} = \ln G_2(w_i, M_i) , \tag{19}$$

where $G_2(w_i, M_i)$ is a homogeneous function of degree μ. Since $h_{j2}^*(p)$ is a homogeneous function of degree zero when $\sigma = 0$, we denote it by $\tilde{h}_{j2}^*(p)$ in order to distinguish it from the case of $\sigma \neq 0$. Eq. (17) becomes:

$$\frac{h_{j1}(p)}{\tilde{h}_{j2}^*(p)} - \frac{h_{j1}(\lambda p)}{\tilde{h}_{j2}^*(\lambda p)} = k(\lambda) = \mu \ln \lambda . \tag{20}$$

Let $H_{j1}(p) = \exp[h_{j1}(p)/h_{j2}(p)]$, then Eq. (20) becomes:

$$H_{j1}(p) = H_{j1}(\lambda p)\lambda^\mu .$$

Hence, $h_{j1}(p)/\tilde{h}_{j2}^*(p) = \ln H_{j1}(p)$, or $h_{j1}(p) = \ln H_{j1}(p)\tilde{h}_{j2}^*(p)$, where $H_{j1}(p)$ is a homogeneous function of degree μ and $\tilde{h}_{j2}^*(p)$ is a homogeneous function of degree zero.

If $\sigma \neq 0$, then differentiating Eq. (16) partially with respect to w_i and M_i, we obtain:

$$\frac{\partial \ell}{\partial w_i}(\lambda w_i, \lambda M_i)\lambda - \lambda^{\sigma}\frac{\partial \ell}{\partial w_i}(w_i, M_i) = 0 ,$$

$$\frac{\partial \ell}{\partial M_i}(\lambda w_i, \lambda M_i)\lambda - \lambda^{\sigma}\frac{\partial \ell}{\partial M_i}(w_i, M_i) = 0 ,$$

which imply that $(\partial \ell/\partial w_i)(w_i, M_i)$ and $(\partial \ell/\partial M_i)(w_i, M_i)$ are both homogeneous functions of degree $(\sigma - 1)$, which in turn implies that $\ell(w_i, M_i)$ is a homogeneous function of degree σ in w_i and M_i, up to the addition of a constant. Thus

$$\ell(w_i, M_i) \equiv \frac{g_2^{**}(w_i, M_i)}{M_i} = G_2^*(w_i, M_i) + D , \tag{21}$$

where $G_2^*(w_i, M_i)$ is a homogeneous function of degree σ and D is a constant. Substituting this back into the individual expenditure function, we obtain:

$$\begin{aligned} e_{ij}(p, w_i, M_i) &= h_{j1}(p)M_i + h_{j2}^*(p)[G_2^*(w_i, M_i)M_i + DM_i] \\ &= h_{j1}^*(p)M_i + h_{j2}^*(p)G_2^{**}(w_i, M_i) , \end{aligned}$$

where $G_2^{**}(w_i, M_i) \equiv G_2^*(w_i, M_i)M_i$, $h_{j1}^*(p) = h_{j1}(p) + h_{j2}^*(p)D$ and $G_2^{**}(w_i, M_i)$ is homogeneous of degree $\sigma + 1$. Homogeneity of degree one of $e_{ij}(p, w_i, M_i)$ in p, w_i, and M_i then implies that $h_{j1}^*(p)$ is homogeneous of degree zero. Thus, we have proved our main result:

Theorem 2. Under the assumption that the system of individual expenditure functions is non-negative, summable, once continuously differentiable and homogeneous of degree one in p, w_i and M_i, a system of aggregate expenditure functions with two indexes ($n = 2$) can be written in the form:

$$\sum_{i=1}^{N} e_{ij}(p, w_i, M_i) = E_j(p, g_1(w_1, \ldots, w_N, M_1, \ldots, M_N), g_2(w_1, \ldots, w_N, M_1, \ldots, M_N)) ,$$
$$j = 1, \ldots, m + 1 ,$$

where the index functions, $g_k(\cdot)$ ($k = 1, 2$), and the aggregate expenditure functions, $E_j(\cdot)$, satisfy assumptions (1)–(4) in Theorem 1, only if the individual expenditure functions take one of the two following forms:

$$e_{ij}(p, w_i, M_i) = \begin{cases} (1)\ h_{j1}^*(p)M_i + h_{j2}^*(p)G_2^{**}(w_i, M_i) , & j = 1, \ldots, m + 1 , \quad \text{or} \\ (2)\ \ln H_{j1}(p)\tilde{h}_{j2}^*(p)M_i + \tilde{h}_{j2}^*(p)M_i \ln G_2(w_i, M_i) , & j = 1, \ldots, m + 1 , \end{cases} \tag{22}$$

where $h_{j1}^*(p)$ and $\tilde{h}_{j2}^*(p)$ are homogeneous of degree zero, $h_{j2}^*(p)$ is homogeneous of degree $-\sigma$, $G_2^{**}(w_i, M_i)$ is homogeneous of degree $\sigma + 1$, $H_{j1}(p)$ is homogeneous of degree $-\mu$ and

$G_2(w_i, M_i)$ is homogeneous of degree μ, $g_2^{**}(w_i, 0) = 0$ and $\lim_{M_i \to 0} M_i \ln G_2(w_i, M_i) = 0$. Furthermore, by Eqs. (10) *and* (11):

$$\sum_{j=1}^{m+1} h_{j1}(p) = 1 , \quad \sum_{j=1}^{m+1} h_{j2}^*(p) = 0 , \quad \sum_{j=1}^{m+1} (\ln H_{j1}(p))\tilde{h}_{j2}^*(p) = 1 , \quad \sum_{j=1}^{m+1} \tilde{h}_{j2}^*(p) = 0 . \tag{23}$$

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